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Row Transfer Matrix Spectra of Cyclic Solid-on-Solid Lattice Models

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The eigenvalue spectra of cyclic solid-on-solid (CSOS) row transfer matrices are studied. An equivalence is established between the inversion identity and the Bethe ansatz functional equations and these equations are solved in the thermodynamic limit by a Wiener–Hopf perturbation technique for the bands of leading excitations. The L-state CSOS model, with crossing parameter $\lambda = \pi s/L$, possesses a $2(L-s)$-fold degenerate largest eigenvalue corresponding to the $2(L-s)$ coexisting phases. The expressions for the largest eigenvalue and free energy coincide with those of the eight-vertex model. The string excitations for $2s < L$ and $2s > L$ admit different classifications and are treated separately. The correlation length is calculated in both regimes, yielding the critical exponent $\nu = L/2s$, in agreement with the scaling relations.

KEY WORDS: CSOS models; Bethe ansatz; correlation length; inversion identities.

1. INTRODUCTION

The cyclic solid-on-solid (CSOS) models[1-3] are a large family of solvable L-state IRF lattice models[4] with crossing parameter $\lambda = \pi s/L$, where L and $s = 1, 2,..., L-1$ are coprime integers. The simplest member of this family, given by $L = 3$, $s = 2$, and $\lambda = 2\pi/3$, correspnds to the three-coloring problem.[5] More generally, the critical CSOS models realize the affine A series in the A-D-E classification.[6-8] In particular, the adjacency graph, giving the allowed states or heights of neighboring spins, is the Dynkin diagram of the affine Lie algebra $A^{(1)}_{L-1}$. The free energy and order

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parameters have been calculated by Pearce and Seaton[2] and yield the critical exponents

$$2-\alpha=L/s \tag{1.1}$$

$$\beta_j=\frac{(L-2j)^2}{4s(L-s)}, \qquad \bar{\beta}_j=\frac{j^2}{s(L-s)}, \qquad j=1,..., \left[\frac{L-1}{2}\right] \tag{1.2}$$

where $[\cdots]$ denotes the integer part. The central charge and operator content of the critical CSOS models have been obtained from finite-size corrections by Kim and Pearce.[9] The central charge is $c=1$ and the dimensions of the scaling operators are of the form

$$x=m^2/2L(L-s)+n^2L(L-s)/8 \tag{1.3}$$

where m and n are integers and m is even for L even. Some of these are related to the order parameter critical exponents in the usual way by

$$x=2\beta/(2-\alpha) \tag{1.4}$$

The critical CSOS models are described by $c=1$ rational conformal field theories[8,10,11] with a radius of compactification given by

$$r^2=p/2p'=\begin{cases} L(L-s)/2, & L \text{ odd} \\ L(L-s)/8, & L \text{ even} \end{cases} \tag{1.5}$$

where p and p' are coprime. The modular invariant partition function is Gaussian and can be decomposed as a sesquilinear form in chiral algebra characters[10]

$$Z(r)=\frac{1}{\eta(q)\,\eta(\bar{q})}\sum_{m,n=-\infty}^{\infty} q^{(1/2)(m/2r+nr)^2}\bar{q}^{(1/2)(m/2r-nr)^2} \tag{1.6}$$

$$\begin{aligned}&=\sum_{a=0}^{p-1}\sum_{b=0}^{p'-1}[\chi_{\mathcal{N},ap'+bp}(q)\,\chi_{\mathcal{N},ap'-bp}(\bar{q})\\ &\quad+\chi_{\mathcal{N},\mathcal{N}+ap'+bp}(q)\,\chi_{\mathcal{N},\mathcal{N}+ap'-bp}(\bar{q})]\end{aligned} \tag{1.7}$$

where q is the modular parameter,

$$\eta(q)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{1.8}$$

is the Dedekind eta function, and the

$$\chi_{\mathcal{N},k}(q)=\frac{1}{\eta(q)}\sum_{n=-\infty}^{\infty} q^{\mathcal{N}(n+k/2\mathcal{N})^2} \tag{1.9}$$

are the level $\mathcal{N} = L(L-s)$ chiral algebra characters. Precisely these characters occur in the expressions for the CSOS local height probabilities.[2] A discussion of the operator algebra of these theories is given in ref. 11.

In this paper we study the eigenvalue spectra of the row transfer matrices of the CSOS models. Such calculations are necessary to obtain correlation lengths and interfacial tensions. Various methods for calculating correlation lengths and interfacial tensions have been introduced. For the eight-vertex model, Baxter[12,13,4] developed a Wiener–Hopf perturbation argument to calculate the bulk free energy and interfacial tension. A related method of integral equations was subsequently used by Johnson *et al.*[14] to calculate the correlation length. More recently, Klümper and Zittartz[15,16] have used the inversion identity[17] along with some analyticity assumptions to classify completely the eigenvalue spectra of both the eight-vertex and *XXZ* models. The Wiener–Hopf perturbation method has also been applied to hard hexagons[18,19] and magnetic hard squares.[20] This method gives the most complete information about the eigenvalues and is the method employed here. In this paper we confine ourselves to the calculation of correlation lengths. The calculation of interfacial tensions will be taken up elsewhere. Our results for the correlation length give the critical exponent

$$\nu = L/2s \tag{1.10}$$

in agreement with the scaling relation

$$2-\alpha = d\nu \tag{1.11}$$

Our calculations in solving the CSOS Bethe ansatz equations have been both guided and confirmed by numerical calculations on finite-size systems using methods developed in ref. 21.

The layout of the paper is as follows. In Section 2, we describe the parametrization of the CSOS models and their row transfer matrices. The inversion identity and Bethe ansatz functional equations are also introduced in this section. The $2(L-s)$ largest eigenvalues and the free energy are calculated in Section 3. The bands of leading excitations for $2s < L$ and $2s > L$ are obtained in Sections 4 and 5, respectively. Formulas for the correlation length and its critical behavior are given in Section 6 and a concluding discussion is given in Section 7. Some miscellaneous technical derivations are collected in appendices.

2. PRELIMINARIES

The spins or heights of the CSOS model occupy the sites of a square lattice and take the integer values 0, 1,..., $L-1$. We will denote spins by a,

b, c, d, etc. All heights are interpreted modulo L with heights of adjacent sites differing by $\pm 1 \bmod L$. In particular, the heights L and 0 are identified so that the heights $L-1$ and 0 are adjacent, as shown in Fig. 1. The constraint on neighboring heights results in the six allowed types of face or vertex configurations shown in Fig. 2.

2.1. Parametrization

The parametrization[2] of the allowed CSOS face weights is given by

$$\alpha_a = W\begin{pmatrix} a-1 & a \\ a & a+1 \end{pmatrix} = W\begin{pmatrix} a+1 & a \\ a & a-1 \end{pmatrix} = \frac{\theta_1(\lambda-u)}{\theta_1(\lambda)} \tag{2.1}$$

$$\begin{aligned}\beta_a &= W\begin{pmatrix} a & a-1 \\ a+1 & a \end{pmatrix} \\ &= W\begin{pmatrix} a & a+1 \\ a-1 & a \end{pmatrix} = \frac{\theta_1(u)[\theta_4(w_{a-1})\,\theta_4(w_{a+1})]^{1/2}}{\theta_1(\lambda)\,\theta_4(w_a)}\end{aligned} \tag{2.2}$$

$$\gamma_a = W\begin{pmatrix} a & a+1 \\ a+1 & a \end{pmatrix} = \frac{\theta_4(w_a+u)}{\theta_4(w_a)} \tag{2.3}$$

$$\delta_a = W\begin{pmatrix} a & a-1 \\ a-1 & a \end{pmatrix} = \frac{\theta_4(w_a-u)}{\theta_4(w_a)} \tag{2.4}$$

where

$$w_a = w_0 + a\lambda \tag{2.5}$$

and

$$\theta_1(u) = \theta_1(u, p) = 2p^{1/4} \sin u \prod_{n=1}^{\infty} (1 - 2p^{2n} \cos 2u + p^{4n})(1 - p^{2n}) \tag{2.6}$$

$$\theta_4(u) = \theta_4(u, p) = \prod_{n=1}^{\infty} (1 - 2p^{2n-1} \cos 2u + p^{4n-2})(1 - p^{2n}) \tag{2.7}$$

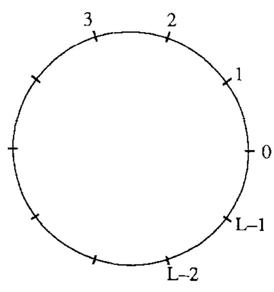

Fig. 1. The Dynkin diagram of the affine Lie algebra $A_{L-1}^{(1)}$. This is the adjacency graph of the CSOS models.

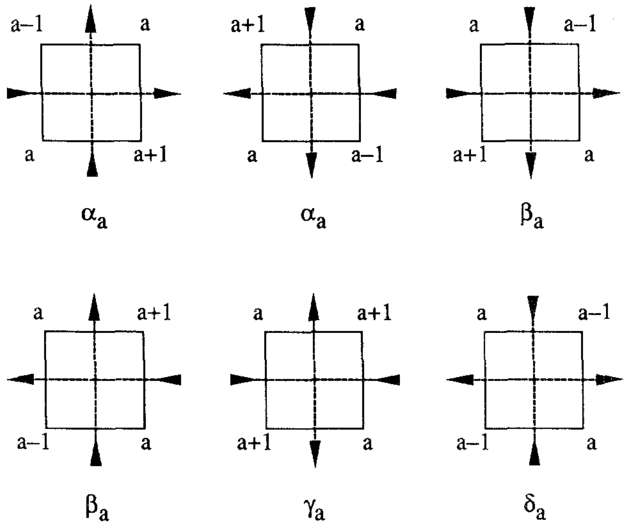

Fig. 2. The six types of allowed face weights for the CSOS models and the corresponding vertex configurations.

are standard elliptic theta functions[22] of nome p. The parameter w_0 is a phase angle interpolating between heights, u is an anisotropy or spectral parameter, and p is a temperature-like variable. The crossing parameter λ is given by

$$\lambda = \pi s/L \tag{2.8}$$

where $s = 1, 2,\dots, L-1$ is an integer coprime to L. The fundamental domain is given by

$$0 \leqslant u < \lambda, \qquad 0 \leqslant w_0 < \pi, \qquad 0 < p < 1 \tag{2.9}$$

In order to study the low-temperature limit, $p \to 1$, we define the elliptic function

$$\begin{aligned} E(z, x) &= \prod_{n=1}^{\infty} (1 - x^{n-1}z)(1 - x^n z^{-1})(1 - x^n) \\ &= \sum_{n=-\infty}^{\infty} (-1)^n x^{n(n-1)/2} z^n \end{aligned} \tag{2.10}$$

Useful properties of this function which we will use repeatedly in the following are

$$E(z, x) = E(x/z, x) = -zE(1/z, x) \tag{2.11}$$

The conjugate modulus forms of the theta functions are

$$\theta_1(u, p) = (\pi/\varepsilon)^{1/2} \exp[-(u-\pi/2)^2/\varepsilon]\ E(e^{-2\pi u/\varepsilon}, p'^2) \tag{2.12}$$

$$\theta_4(u, p) = (\pi/\varepsilon)^{1/2} \exp[-(u-\pi/2)^2/\varepsilon]\ E(-e^{-2\pi u/\varepsilon}, p'^2) \tag{2.13}$$

where the conjugate nome is given by

$$p = \exp(-\varepsilon), \qquad p' = \exp(-\pi^2/\varepsilon) \tag{2.14}$$

Defining the variables

$$w = \exp(-2\pi u/\varepsilon), \qquad v = \exp(-2\pi w_0/\varepsilon), \qquad x = \exp(-\pi^2/L\varepsilon) \tag{2.15}$$

the face weights transform to (up to a proportionality constant which we neglect here)

$$\alpha_a = w^{(L-s)/2L} \frac{E(x^{2s}/w)}{E(x^{2s})} \tag{2.16}$$

$$\beta_a = x^s w^{(s-L)/2L} \frac{E(w)[E(-vx^{2s(a+1)})\ E(-vx^{2s(a-1)})]^{1/2}}{E(x^{2s})\ E(-vx^{2sa})} \tag{2.17}$$

$$\gamma_a = w^{(s-L)/2L + sa/L + w_0/\pi} \frac{E(-vx^{2sa}w)}{E(-vx^{2sa})} \tag{2.18}$$

$$\delta_a = w^{(s+L)/2L - sa/L - w_0/\pi} \frac{E(-vx^{2sa}/w)}{E(-vx^{2sa})} \tag{2.19}$$

where

$$E(z) = Z(z, x^{2L}) \tag{2.20}$$

2.2. Row Transfer Matrix

The row transfer matrix $\mathbf{V}$ for the CSOS models is defined via its elements as

$$V(\mathbf{a} \mid \mathbf{b}) = \prod_{j=1}^{N} W\begin{pmatrix} b_j & b_{j+1} \\ a_j & a_{j+1} \end{pmatrix} \tag{2.21}$$

where $\mathbf{a} = (a_1, ..., a_N)$ and $\mathbf{b} = (b_1, ..., b_N)$ are the configurations of two adjacent rows of heights. We impose periodic boundary conditions on the heights, i.e., $a_{N+1} = a_1$ and $b_{N+1} = b_1$. Following ref. 9, it is convenient to represent a given row configuration $\mathbf{a}$ by $(a_1, \boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_N)$ and $\sigma_i = a_{i+1} - a_i = \pm 1$ is the difference of adjacent heights with $\pm(L-1)$

interpreted as ∓ 1. Thus, σ_i represents the corresponding arrow configuration between adjacent heights, with $\sigma_i = 1$ (-1) for an up (down) arrow on the ith vertical bond. The arrow configurations associated with the allowed face weights are shown in Fig. 2.

Because of the six-vertex constraints, the number of up and down arrows in each row of vertical bonds is a conserved quantity and subsequently **V** breaks up into disjoint sectors. Specifically, the operator **Q** defined by

$$Q(\mathbf{a} \mid \mathbf{b}) = \frac{1}{2} \sum_{i=1}^{N} \sigma_i \prod_{j=1}^{N} \delta(a_j, b_j) \tag{2.22}$$

commutes with **V**. Hence its eigenvalues are good quantum numbers, labeling the sectors. Letting n be the number of down arrows in a row, the "charge" Q is given by

$$2Q = \sum_{i=1}^{N} \sigma_i = N - 2n = \gamma L \tag{2.23}$$

where $\gamma = 0, \pm 1, \ldots$ is the winding number. It represents the number of times a given height configuration cycles or "winds" through all allowed heights. The allowed values of γ are dependent on N and L:

$$\begin{aligned} &\gamma \in Z && \text{for } N \text{ even and } L \text{ even} \\ &\gamma \in 2Z && \text{for } N \text{ even and } L \text{ odd} \\ &\gamma \in 2Z + 1 && \text{for } N \text{ odd and } L \text{ odd} \end{aligned} \tag{2.24}$$

For the remaining case, N odd and L even, there are no configurations compatible with the periodic boundary condition.

For a given value of L, the number of possible configurations along a periodic row of N sites, $\mathcal{N}(N, L)$, can be derived from the properties of the $L \times L$ adjacency matrix **A**. Its elements are defined by

$$A_{ab} = \begin{cases} 1 & \text{if} \quad a \text{ and } b \text{ are allowed neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{2.25}$$

and reflect the adjacency condition on the nearest-neighbor heights. The number of allowed configurations is given by

$$\mathcal{N}(N, L) = \operatorname{Tr} \mathbf{A}^N = \sum_a \Lambda_a^N \tag{2.26}$$

The eigenvalues Λ_a of the adjacency matrix are given by

$$\Lambda_a = 2 \cos(2\pi a/L), \qquad a = 0, 1, \ldots, L-1 \tag{2.27}$$

With this result we have

$$\mathcal{N}(N, L) = L \sum_{k=0}^{N}{}^{*} \binom{N}{k} \tag{2.28}$$

where $\binom{m}{n}$ is the binomial coefficient and the starred sum is restricted to k satisfying $N-2k=0 \pmod{L}$. As an example, we find $\mathcal{N}(8,4)=512$. Among these states, 280 states have winding number $\gamma=0$ with the remaining states shared equally between $\gamma=\pm 1$. In contrast, for free boundary conditions on the heights, the sum in (2.28) is unconstrained and there are $L \cdot 2^N$ states.

2.3. Low-Temperature Limit

Much insight into the structure of the eigenvalue spectrum of the CSOS row transfer matrices is to be gained by considering the low-temperature limit. In this limit, $x \to 0$ with $w \sim 1$, the vertex weights (2.16)–(2.19) have the leading-order behavior

$$\alpha_a = w^{(L-s)/2L} \tag{2.29}$$

$$\beta_a = 0 \tag{2.30}$$

$$\gamma_a = w^{s/2L + sa/L + w_0/L - [sa/L]} \tag{2.31}$$

$$\delta_a = w^{s/2L - sa/L - w_0/L + [sa/L]} \tag{2.32}$$

where $[\cdots]$ denotes the integer part. This simplification of the weights is indicative of the inherent "band" structure in the eigenvalue spectrum of the transfer matrix $\mathbf{V}$. In the low-temperature limit, the only non-vanishing elements of $\mathbf{V}$ are those for which the adjacent rows of heights in (2.21) are related by a simple shift, that is, $\mathbf{a}=(a_1,..., a_N)$ and $\mathbf{b}=(a_N, a_1, a_2,..., a_{N-1})$. This gives a way of enumerating the number of levels in a given band. To make this clear, let us consider the case $N=6$ with $L=3$ and $s=1$ or 2 as a specific example. For each of the $\mathcal{N}(6,3)=60$ possible height configurations along a row, the corresponding element of the transfer matrix is a power of w, with the exponent labeling the band. The number of states in each band is indicated for this example in Table I.

Unlike the eight-vertex model, which has two numerically largest eigenvalues of opposite sign, the CSOS model has $2(L-s)$ largest eigenvalues corresponding to $2(L-s)$ coexisting phases. All the other bands have of order N or more eigenvalues which become continuous in the thermodynamic limit. The nature of the $2(L-s)$ ground states has been discussed in ref. 2. For the same examples as in Table I, we show in Fig. 3 a

Table I. Eigenvalue Bands for $N=6$

Band	$L=3,\ s=1$	$L=3,\ s=2$
1	4	2
w	30	24
w^2	24	30
w^3	2	4

physical picture of the ground states and the elementary excitations. In Fig. 3a, (i) and (ii) represent the two ground states for $L=3$ and $s=2$, while (iii) indicates a typical elementary excitation. Both the excitations in (iii) and (iv) belong to the leading band of excitations. We see that each of the excitations shown can occur in either of six places for each ground state, giving the total of 24 excitations in the w band. On the other hand, the situation becomes more complicated for $L=3$ and $s=1$, for which the four ground states are indicated in Fig. 3b. In this case the elementary excitation of (iii) can cross over to another of the ground-state configurations. This effective "tunneling" between ground states makes the counting of excitations more difficult.

2.4. Inversion Identity and Bethe Ansatz

The commuting row transfer matrices of solvable IRF models have been found to satisfy special functional equations given by the Bethe ansatz[23] or inversion identities.[24,17,25] The derivation[26] of the inversion

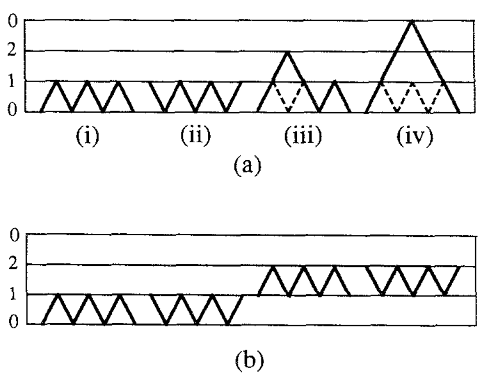

Fig. 3. Physical picture of the ground states of the CSOS model for $N=6$, $L=3$ and (a) $s=2$, (b) $s=1$. (a)(iii) An elementary excitation; (a)(iv) A composite excitation.

identity for the CSOS models with periodic boundary conditions is given in Appendix A. The inversion identity takes the form

$$\mathbf{V}(u)\,\mathbf{V}(u+\lambda)=\phi(\lambda-u)\,\phi(\lambda+u)\mathbf{I}+\phi(u)\,\mathbf{P}(u) \tag{2.33}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{P}(u)$ is an auxiliary matrix that commutes with $\mathbf{V}(u)$, and $[\mathbf{V}(u),\mathbf{V}(v)]=0$ for all u and v. The function $\phi(u)$ is defined by

$$\phi(u)=\left[\frac{\theta_1(u)}{\theta_1(\lambda)}\right]^N \tag{2.34}$$

The corresponding functional equation deriving from the Bethe ansatz is

$$\mathbf{V}(u)\,\mathbf{Q}(u)=\phi(\lambda-u)\,\mathbf{Q}(u+\lambda)+\phi(u)\,\mathbf{Q}(u-\lambda) \tag{2.35}$$

where $[\mathbf{Q}(u),\mathbf{Q}(v)]=0$ for all u and v and $\mathbf{Q}(u)$ is another family of matrices that commutes with $\mathbf{V}(u)$. These functional equations are formally identical to the functional equations satisfied by the row transfer matrix of the eight-vertex model. In the context of the CSOS models, however, these equations are subject to different supplementary conditions and admit more solutions corresponding to the larger dimension of the transfer matrix.

The inversion identity (2.33) and the Bethe ansatz equation (2.35) each separately contain enough information to solve for the complete eigenvalue spectrum of the transfer matrix $\mathbf{V}(u)$. In this paper, we elect to calculate such quantities as the free energy and the correlation length via the more familiar Bethe ansatz equation (2.35). For all intents and purposes, however, the two equations are equivalent. Reshetikhin[27] has given an analytic ansatz for obtaining Bethe ansatz equations from inversion identities. We apply this to the CSOS models in Appendix B. Conversely, assuming $\mathbf{Q}(u)$ is invertible, we can multiply $\mathbf{V}(u)$ and $\mathbf{V}(u+\lambda)$ as given by the Bethe ansatz equation (2.35). This yields a functional equation of the form of the inversion identity (2.33) with $\mathbf{P}(u)$ explicitly related to $\mathbf{Q}(u)$.

Both the inversion identity and the Bethe ansatz equation are independent of the phase angle w_0. Since all the eigenvalues of $\mathbf{V}(u)$ occur among the solutions to these functional equations, this implies that, although the eigenvalues depend on the spectral parameter u, they must also be independent of w_0. Given the definitions of the vertex weights (2.16)–(2.19) and their explicit dependence on the variable w_0, this is not at all obvious. However, this remarkable fact is indeed confirmed by direct numerical diagonalization of the transfer matrix. In contrast, the eigenvectors of $\mathbf{V}(u)$ are independent of u, but do depend on the phase angle w_0.

For convenience, we will assume in the sequel that the number of sites N is a multiple of L. The transfer matrix then satisfies the crossing and quasiperiodicity properties

$$\mathbf{V}^{\dagger}(u)=\mathbf{V}(\lambda-u) \tag{2.36}$$

$$\mathbf{V}(u+\pi)=(-1)^{N}\,\mathbf{V}(u) \tag{2.37}$$

$$\mathbf{V}(u+\pi\tau)=(-p^{-1}e^{-2iu})^{N}\,e^{2ni\lambda}\mathbf{V}(u) \tag{2.38}$$

where τ is related to the nome p by $p=e^{\pi i\tau}$.

Introducing the variables

$$u=\tfrac{1}{2}\lambda+v, \qquad s_j=\tfrac{1}{2}\lambda+v_j \tag{2.39}$$

and factoring $\mathbf{Q}$ as in Appendix B, the Bethe ansatz equation for the eigenvalues of $\mathbf{V}$ becomes

$$V(v)=\phi\left(\frac{\lambda}{2}-v\right)\prod_{j=1}^{n}\frac{\theta_1(v-v_j+\lambda)}{\theta_1(v-v_j)}+\phi\left(\frac{\lambda}{2}+v\right)\prod_{j=1}^{n}\frac{\theta_1(v-v_j-\lambda)}{\theta_1(v-v_j)} \tag{2.40}$$

Here the equations determining the zeros v_j of $\mathbf{Q}$ take the form

$$\frac{\phi(\lambda/2-v_k)}{\phi(\lambda/2+v_k)}=-\prod_{j=1}^{n}\frac{\theta_1(v_k-v_j-\lambda)}{\theta_1(v_k-v_j+\lambda)}, \qquad k=1,\ldots,n \tag{2.41}$$

To use these equations in the low-temperature ordered limit, we again need to apply the conjugate modulus transformations. Defining the variables

$$z=e^{-2\pi v/\varepsilon}, \qquad z_j=e^{-2\pi v_j/\varepsilon} \tag{2.42}$$

we obtain the low-temperature form of the Bethe ansatz equations

$$\begin{aligned}V(z)=(-1)^{n}\,x^{\gamma(L-s)/2}&\left\{z^{\gamma(L-s)/2}\left[\frac{E(x^{s}/z)}{E(x^{2s})}\right]^{N}\prod_{j=1}^{n}z_j^{(L-s)/L}\frac{E(x^{2s}z/z_j)}{E(z_j/z)}\right.\\&\left.+\,z^{-\gamma(L-s)/2}\left[\frac{E(x^{s}z)}{E(x^{2s})}\right]^{N}\prod_{j=1}^{n}z_j^{-(L-s)/L}\frac{E(x^{2s}z_j/z)}{E(z/z_j)}\right\}\end{aligned} \tag{2.43}$$

with

$$z_k^{n+\gamma(L-s)}\left[\frac{E(x^{s}/z_k)}{E(x^{s}z_k)}\right]^{N}+(-1)^{n}\prod_{j=1}^{n}z_j^{(2s-L)/L}\frac{E(x^{2s}z_j/z_k)}{E(x^{2s}z_k/z_j)}=0 \tag{2.44}$$

for $k=1,\ldots,n$. These are our key equations.

A useful consequence of the crossing property (2.36) is that it reflects the crossing symmetry of the eigenspectrum. In terms of the variables defined in (2.39) and (2.42), we have $\mathbf{V}^\dagger(z)=\mathbf{V}(1/z)$, implying that the eigenspectrum for $|z|<1$ is related by complex conjugation to that for $|z|>1$. One further point also of use in the following is that at $w=1$ $(u=0)$ the transfer matrix $\mathbf{V}$ reduces to a shift operator with eigenvalues the Nth roots of unity, as can readily be seen from the definition of the vertex weights in (2.1)–(2.4).

3. LARGEST EIGENVALUE

In this section we evaluate the largest eigenvalue of the row transfer matrix $\mathbf{V}$ in the thermodynamic limit by applying the perturbation technique first developed by Baxter for the eight-vertex model.[12,4] We begin by considering the low-temperature limit $x\to 0$ where the Bethe ansatz equations (2.44) reduce to

$$z^n+(-1)^n(z_1\cdots z_n)^{(2s-L)/L}=0 \tag{3.1}$$

with $z\equiv z_k$, $k=1,\dots,n$. To be compatible with the ground states, we have chosen N to be even. The largest eigenvalues then occur in the charge $Q=0$ sector, i.e., with $\gamma=0$ and $n=N/2$. Arguing as in Baxter, we set

$$\prod_{j=1}^{n}(z-z_j)\equiv \text{l.h.s. of (3.1)} \tag{3.2}$$

Thus we must have

$$(z_1\cdots z_n)=(z_1\cdots z_n)^{(2s-L)/L} \tag{3.3}$$

and subsequently

$$(z_1\cdots z_n)^{2(L-s)/L}=1 \tag{3.4}$$

Hence (3.1) admits $2(L-s)$ solutions consisting of the n roots of the equations

$$z^n+(-1)^n e^{\pi ik(2s-L)/(L-s)}=0,\qquad k=0,1,\dots,2(L-s)-1 \tag{3.5}$$

In this limit the eigenvalue expression (2.43) reduces to

$$V_0(z)=(-)^n\left[\prod_{j=1}^{n}\frac{z_j^{(L-s)/L}}{1-z_j/z}+\prod_{j=1}^{n}\frac{z_j^{(s-L)/L}}{1-z/z_j}\right] \tag{3.6}$$

which, from (3.1), (3.2), and (3.4), yields

$$V_0(z)=\pm 1 \tag{3.7}$$

each sign occurring $L-s$ times. This gives the $2(L-s)$ largest eigenvalues, each corresponding to a choice of the phase factor in (3.5).

3.1. Perturbation Argument

We now consider the perturbation expansion about the low-temperature solution for large N and $x<1$. To do this we define the functions[12,4]

$$A(z)=\prod_{k=0}^{\infty}(1-x^{2Lk}z)^N, \qquad B(1/z)=\prod_{k=1}^{\infty}(1-x^{2Lk}/z)^N \tag{3.8}$$

$$F(z)=\prod_{j=1}^{n}\prod_{k=0}^{\infty}(1-x^{2Lk}z/z_j), \qquad G(1/z)=\prod_{j=1}^{n}\prod_{k=1}^{\infty}(1-x^{2Lk}z_j/z) \tag{3.9}$$

Here the functions $A(z)$ and $B(1/z)$ are known and $F(z)$ and $G(1/z)$, which depend on the zeros z_j, are to be determined. With these definitions the eigenvalue equation (2.43) can be written as

$$V_0(z)=V_0^{(l)}(z)+V_0^{(r)}(z) \tag{3.10}$$

where we have set

$$V_0^{(l)}(z)=(-1)^n\,(z_1\cdots z_n)^{(L-s)/L} \times\frac{A(x^{2L-s}z)\,B(1/x^{2L-s}z)\,F(x^{2s}z)\,G(1/x^{2s}z)}{A(x^{2s})\,B(1/x^{2s})\,F(x^{2L}z)\,G(1/x^{2L}z)} \tag{3.11}$$

and

$$V_0^{(r)}(z)=(-1)^n\,(z_1\cdots z_n)^{(s-L)/L} \times\frac{A(x^{s}z)\,B(1/x^{s}z)\,F(x^{2(L-s)}z)\,G(1/x^{2(L-s)}z)}{A(x^{2s})\,B(1/x^{2s})\,F(z)\,G(1/z)} \tag{3.12}$$

Depending on the sign of v, one of the two terms becomes exponentially small in the thermodynamic limit. Neglecting these terms in the thermodynamic limit yields

$$V_0(z)=\begin{cases}V_0^{(l)}(z), & \text{for } z>1\\ V_0^{(r)}(z), & \text{for } z<1\end{cases} \tag{3.13}$$

On the other hand, the Bethe ansatz equations (2.44) are

$$z^n\frac{B(1/x^{2L-s}z)\,G(1/x^{2s}z)}{B(1/x^{s}z)\,G(1/x^{2(L-s)}z)}+(-1)^n\,(z_1\cdots z_n)^{(2s-L)/L}\frac{A(x^{s}z)\,F(x^{2(L-s)}z)}{A(x^{2L-s}z)\,F(x^{2s}z)}=0 \tag{3.14}$$

At this point we write

$$\prod_{j=1}^{n} (z - z_j) \equiv \text{l.h.s. of } (3.14) \tag{3.15}$$

Thus, in the region $|z| < 1$ we have, to exponentially small terms [e.g., $O(\delta^N)$ with $0 \leqslant \delta < 1$],

$$\frac{A(x^s z)\, F(x^{2(L-s)} z)}{A(x^{2L-s} z)\, F(x^{2s} z)} = \prod_{j=1}^{n} \left(1 - \frac{z}{z_j}\right) = \frac{F(z)}{F(x^{2L} z)} \tag{3.16}$$

while for $|z| > 1$

$$\frac{B(1/x^{2L-s} z)\, G(1/x^{2s} z)}{B(1/x^s z)\, G(1/x^{2(L-s)} z)} = \prod_{j=1}^{n} \left(1 - \frac{z_j}{z}\right) = \frac{G(1/x^{2L} z)}{G(1/z)} \tag{3.17}$$

These last two equations can be solved by recursion,[12] with result

$$F(z) = \prod_{m=0}^{\infty} \frac{A(x^{(4m+1)s} z)}{A(x^{(4m+3)s} z)} \tag{3.18}$$

$$G\left(\frac{1}{z}\right) = \prod_{m=0}^{\infty} \frac{B(x^{(4m+1)s}/z)}{B(x^{(4m+3)s}/z)} \tag{3.19}$$

Substitution of these results into (3.11) and (3.12) then gives the largest eigenvalue in the form

$$\begin{aligned} V_0^{(l)}(z) &= (-1)^n\, (z_1 \cdots z_n)^{(L-s)/L}\, \frac{A(x^{2L-s} z)\, B(1/x^{2L-s} z)}{A(x^{2s})\, B(1/x^{2s})} \\ &\times \prod_{m=0}^{\infty} \frac{A(x^{(4m+3)s} z)}{A(x^{(4m+5)s} z)} \frac{A(x^{(4m+3)s+2L} z)\, B(x^{(4m-1)s}/z)\, B(x^{(4m+3)s-2L}/z)}{A(x^{(4m+1)s+2L} z)\, B(x^{(4m+1)s}/z)\, B(x^{(4m+1)s-2L}/z)} \end{aligned} \tag{3.20}$$

$$\begin{aligned} V_0^{(r)}(z) &= (-1)^n\, (z_1 \cdots z_n)^{(s-L)/L}\, \frac{A(x^s z)\, B(1/x^s z)}{A(x^{2s})\, B(1/x^{2s})} \\ &\times \prod_{m=0}^{\infty} \frac{A(x^{(4m-1)s+2L} z)}{A(x^{(4m+1)s+2L} z)} \frac{A(x^{(4m+3)s} z)\, B(x^{(4m+3)s-2L}/z)\, B(x^{(4m+3)s}/z)}{A(x^{(4m+1)s} z)\, B(x^{(4m+5)s-2L}/z)\, B(x^{(4m+1)s}/z)} \end{aligned} \tag{3.21}$$

The next step in the argument is to realize that these last two rather cumbersome expressions can be written in a far simpler form. Some details of the proofs of the required identities are described in Appendix C. The final result is valid for both $z < 1$ and $z > 1$ provided the roots z_j still satisfy the condition (3.4), which was obtained in the low-temperature limit. However, this result can be seen to hold more generally from the identification (3.15) and by setting $z=0$ in (3.14). Explicitly, we thus have for the free energy per site

$$\lim_{N\to\infty} N^{-1}\log V_0(w)=\mathscr{S}(w, x^{2s}, x^{2L})=\log \mathscr{T}(w, x^{2s}, x^{2L}) \qquad (3.22)$$

where $w=x^s z$,

$$\mathscr{S}(w, y, p)=\sum_{n=1}^{\infty}\frac{(1-w^n)(1-y^n w^{-n})(y^n+y^{-n}p^n)}{n(1-p^n)(1+y^n)} \qquad (3.23)$$

and

$$\begin{aligned}\mathscr{T}(w, y, p)&=\exp\mathscr{S}(w, y, p)\\ &=\frac{Q(p)}{E(y, p)}\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}\left[\frac{(1-p^{m-1}y^{2n-1}w)(1-p^m y^{2n-3}w)}{(1-p^{m-1}y^{2n}w)(1-p^m y^{2n-2}w)}\right.\\ &\quad\left.\times\frac{(1-p^{m-1}y^{2n}w^{-1})(1-p^m y^{2n-2}w^{-1})}{(1-p^{m-1}y^{2n+1}w^{-1})(1-p^m y^{2n-1}w^{-1})}\right]\end{aligned} \qquad (3.24)$$

with

$$Q(z)=\prod_{n=1}^{\infty}(1-z^n) \qquad (3.25)$$

The function $\mathscr{S}$ is precisely the function appearing in the expression for the largest eigenvalue of the eight-vertex model.[12,4] In particular, we note that (3.22) is the result already obtained in ref. 2 using the "inversion relation trick," which relies on certain analyticity assumptions which cannot be established *a priori*.

4. LEADING EXCITATIONS ($2s<L$)

In this and the following section we calculate the excitation spectrum of the CSOS row transfer matrices. As for the eight-vertex model,[14–16] we find that the nature of the excitations differs in two fundamental regimes determined by the inequalities $2s<L$ and $2s>L$. In this section we consider the case $2s<L$. In this case, we find the band of next-largest eigenvalues is composed of elementary 1-string and 2-string excitations. Throughout this and the remaining sections we assume that N is even and the winding number is $\gamma=0$.

4.1. 1-Strings

As for the calculation of the largest eigenvalue, it is instructive to begin with the low-temperature limit. For each of the largest eigenvalues, we saw that the $n = N/2$ zeros lie on the unit circle. For the 1-string excitations, one of the zeros is excited to the circle $|z| = x^L$. We write the zeros as

$$z_j = a_j \qquad \text{for} \quad j = 1, \ldots, n-1 \tag{4.1}$$

$$z_n = bx^L \tag{4.2}$$

with $|a| \sim |b| \sim 1$. The Bethe ansatz equations (2.44) can then be written

$$a^{n+1}\left[\frac{E(x^s/a)}{E(x^s a)}\right]^N + (-1)^{n+1}\, b(A_{n-1}b)^{(2s-L)/L} \times \frac{E(x^{L-2s}a/b)}{E(x^{L-2s}b/a)} \prod_{j=1}^{n-1} \frac{E(x^{2s}a_j/a)}{E(x^{2s}a/a_j)} = 0 \tag{4.3}$$

$$\left[\frac{E(x^{L-s}b)}{E(x^{L-s}/b)}\right]^N = (A_{n-1}b)^{2s/L} \prod_{j=1}^{n-1} \frac{E(x^{L-2s}b/a_j)}{E(x^{L-2s}a_j/b)} \tag{4.4}$$

Here we have set $a \equiv a_k$, $k = 1, \ldots, n-1$, and defined

$$A_m = \prod_{j=1}^{m} a_j \tag{4.5}$$

In the low-temperature limit, $x \to 0$, we have

$$a^{n+1} + (-1)^{n+1}\, (A_{n-1}b)^{2s/L} A_{n-1}^{-1} = 0 \tag{4.6}$$

$$(A_{n-1}b)^{2s/L} = 1 \tag{4.7}$$

Combining these equations thus yields

$$a^{n+1} + (-1)^{n+1} A_{n-1}^{-1} = 0 \tag{4.8}$$

from which we readily establish

$$A_{n-1}^N = 1 \tag{4.9}$$

Now (4.8) is to be solved for $n-1$ unknowns, yet the equation is of order $n+1$. We see then there are effectively the two zeros, a_n and a_{n+1}, left over—they will turn out to be "holes." For this case we write

$$\prod_{j=1}^{n+1} (a - a_j) = \text{l.h.s. of } (4.8) \tag{4.10}$$

which yields $A_{n+1} = A_{n-1}^{-1}$, i.e., the holes a_n and a_{n+1} must satisfy the constraint

$$a_n a_{n+1} = A_{n-1}^{-2} \tag{4.11}$$

As a typical example, let us consider the values $L = 3$ and $s = 1$ with $N = 6$, i.e., with $n = 3$ zeros. From (4.8) and (4.9), we have

$$a^4 + A_2^{-1} = 0 \qquad \text{with} \quad A_2 = \pm 1, e^{\pm \pi i/3}, e^{\pm 2\pi i/3} \tag{4.12}$$

The possible values of b follow from (4.7), which is simply $A_2 b = \pm 1$. Finally, from (4.11) we select the solutions a_1 and a_2 of (4.12) with a_3 and a_4 satisfying $a_3 a_4 = A_2^{-2}$. In this way we arrive at the 18 allowed configurations shown in Fig. 4. In each case, the eigenvalue lies in the w band, that is, it satisfies

$$|V_1(z)| = w \tag{4.13}$$

To proceed with the perturbation argument, we define the functions

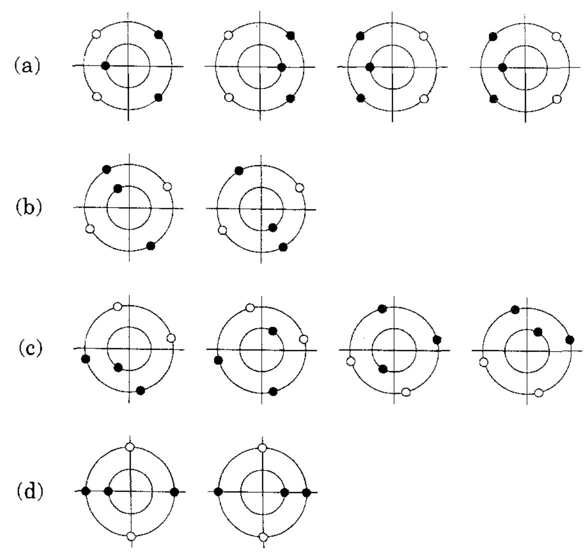

Fig. 4. Schematic representation of the 1-string Bethe ansatz zeros in the low-temperature limit for $N = 6$, $L = 3$, and $s = 2$, with (a) $A_2 = 1$, (b) $A_2 = \exp(\pi i/3)$, (c) $A_2 = \exp(2\pi i/3)$, (d) $A_2 = -1$. Here A_2 is the product of the two zeros on the unit (outer) circle on which the holes are denoted by open circles. The inner circle is of radius $|z| = x^L$. The distribution of zeros for the remaining cases, $A_2 = \exp(-2\pi i/3)$ and $A_2 = \exp(-\pi i/3)$, are conjugate to (c) and (b), respectively.

$A(z)$ and $B(1/z)$ as in (3.8). However, the functions $F(z)$ and $G(1/z)$ in (3.9) are to be replaced by

$$F_1(z) = \prod_{j=1}^{n+1} \prod_{k=0}^{\infty} (1 - x^{2Lk} z/a_j), \qquad G_1(1/z) = \prod_{j=1}^{n+1} \prod_{k=1}^{\infty} (1 - x^{2Lk} a_j/z) \tag{4.14}$$

In addition, we define the extra functions

$$X(z) = \prod_{k=0}^{\infty} (1 - x^{2Lk} z/b), \qquad Y(1/z) = \prod_{k=1}^{\infty} (1 - x^{2Lk} b/z) \tag{4.15}$$

$$R_1(z) = \prod_{j=n}^{n+1} \prod_{k=0}^{\infty} (1 - x^{2Lk} z/a_j), \qquad S_1(1/z) = \prod_{j=n}^{n+1} \prod_{k=1}^{\infty} (1 - x^{2Lk} a_j/z) \tag{4.16}$$

Like the functions $A(z)$ and $B(1/z)$, these latter functions are treated as known, and the functions $F_1(z)$ and $G_1(1/z)$ are to be determined. With these definitions, the crucial equations (4.3) and (4.4) can be written as

$$\begin{aligned} &a^{n+1} \frac{B(1/x^{2L-s}a)\, G_1(1/x^{2s}a)\, Y(1/x^{L+2s}a)\, S_1(1/x^{2(L-s)}a)}{B(1/x^{s}a)\, G_1(1/x^{2(L-s)}a)\, Y(1/x^{L-2s}a)\, S_1(1/x^{2s}a)} \\ &\quad + (-1)^{n+1} b (A_{n-1} b)^{(2s-L)/L} \\ &\quad \times \frac{A(x^s a)\, F_1(x^{2(L-s)}a)\, X(x^{L-2s}a)\, R_1(x^{2s}a)}{A(x^{2L-s}a)\, F_1(x^{2s}a)\, X(x^{L+2s}a)\, R_1(x^{2(L-s)}a)} = 0 \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} (A_{n-1} b)^{2s/L} &= \frac{A(x^{L-s}b)\, B(1/x^{L-s}b)}{A(x^{L+s}b)\, B(1/x^{L+s}b)} \\ &\quad \times \frac{F_1(x^{L+2s}b)\, G_1(1/x^{L+2s}b)\, R_1(x^{L-2s}b)\, S_1(1/x^{L-2s}b)}{F_1(x^{L-2s}b)\, G_1(1/x^{L-2s}b)\, R_1(x^{L+2s}b)\, S_1(1/x^{L+2s}b)} \end{aligned} \tag{4.18}$$

We begin by considering (4.17). As in (4.10), we write

$$\prod_{j=1}^{n+1} (a - a_j) = \text{l.h.s. of } (4.17) \tag{4.19}$$

Equating the dominant terms in this equation for $|a| < 1$ and $|a| > 1$ yields

$$\begin{aligned} &\frac{A(x^s a)\, F_1(x^{2(L-s)}a)\, X(x^{L-2s}a)\, R_1(x^{2s}a)}{A(x^{2L-s}a)\, F_1(x^{2s}a)\, X(x^{L+2s}a)\, R_1(x^{2(L-s)}a)} \\ &\quad = \prod_{j=1}^{n+1} \left(1 - \frac{a}{a_j}\right) = \frac{F_1(a)}{F_1(x^{2L}a)} \end{aligned} \tag{4.20}$$

and

$$\frac{B(1/x^{2L-s}a)\,G_1(1/x^{2s}a)\,Y(1/x^{L+2s}a)\,S_1(1/x^{2(L-s)}a)}{B(1/x^{s}a)\,G_1(1/x^{2(L-s)}a)\,Y(1/x^{L-2s}a)\,S_1(1/x^{2s}a)}$$

$$=\prod_{j=1}^{n+1}\left(1-\frac{a_j}{a}\right)=\frac{G_1(1/x^{2L}a)}{G_1(1/a)} \tag{4.21}$$

In order to solve these relations for $F_1(a)$ and $G_1(1/a)$, we introduce the functions

$$\tilde{F}_1(a)=\frac{F_1(a)}{F_1(x^{2(L-s)}a)}, \qquad \tilde{G}_1\left(\frac{1}{a}\right)=\frac{G_1(1/a)}{G_1(1/x^{2(L-s)}a)} \tag{4.22}$$

and write (4.20) and (4.21) in the form

$$\tilde{F}_1(a)=\frac{A(x^{s}a)\,X(x^{L-2s}a)\,R_1(x^{2s}a)}{A(x^{2L-s}a)\,X(x^{L+2s}a)\,R_1(x^{2(L-s)}a)}\,\frac{1}{\tilde{F}_1(x^{2s}a)} \tag{4.23}$$

$$\tilde{G}_1\left(\frac{1}{a}\right)=\frac{B(x^{s}/a)\,Y(x^{4s-L}/a)\,S_1(1/a)}{B(x^{3s-2L}/a)\,Y(x^{-L}/a)\,S_1(x^{4s-2L}/a)}\,\frac{1}{\tilde{G}_1(1/x^{2s}a)} \tag{4.24}$$

These equations can then be solved by recursion, with result

$$\tilde{F}_1(a)=\frac{X(x^{L-2s}a)}{X(x^{L}a)}\prod_{m=0}^{\infty}\frac{A(x^{(4m+1)s}a)\,A(x^{(4m+1)s+2L}a)}{A(x^{(4m+3)s}a)\,A(x^{(4m-1)s+2L}a}$$

$$\times\frac{R_1(x^{(4m+2)s}a)\,R_1(x^{4ms+2L}a)}{R_1(x^{(4m+4)s}a)\,R_1(x^{(4m-2)s+2L}a)} \tag{4.25}$$

and

$$\tilde{G}_1\left(\frac{1}{a}\right)=\frac{Y(x^{2s-L}/a)}{Y(x^{-L}/a)}\prod_{m=0}^{\infty}\frac{B(x^{(4m+1)s}/a)\,B(x^{(4m+5)s-2L}/a)}{B(x^{(4m+3)s}/a)\,B(x^{(4m+3)s-2L}/a)}$$

$$\times\frac{S_1(x^{4ms}/a)\,S_1(x^{(4m+6)s-2L}/a)}{S_1(x^{(4m+2)s}/a)\,S_1(x^{(4m+4)s-2L}/a)} \tag{4.26}$$

From (4.22) we thus obtain the solutions

$$F_1(a)=\prod_{n=0}^{\infty}\tilde{F}_1(x^{2n(L-s)}a) \tag{4.27}$$

$$G_1(1/a)=\prod_{n=0}^{\infty}\tilde{G}_1(1/x^{2n(L-s)}a) \tag{4.28}$$

Turning to the eigenvalue expression (2.43), the analogous result to (3.13) for $V_0(z)$ is

$$V_1(z) = \begin{cases} V_1^{(l)}(z) & \text{for} \quad z > 1 \\ V_1^{(r)}(z) & \text{for} \quad z < 1 \end{cases} \tag{4.29}$$

with

$$\begin{aligned} V_1^{(l)}(z) = & -(-1)^n (A_{n-1} b)^{(L-s)/L} \left(\frac{x^s z}{b}\right) \frac{A(x^{2L-s}z)\, B(1/x^{2L-s}z)}{A(x^{2s})\, B(1/x^{2s})} \\ & \times \frac{X(x^{2s+L}z)\, Y(1/x^{2s+L}z)}{X(x^L z)\, Y(1/x^L z)} \\ & \times \frac{R_1(x^{2L}z)\, S_1(1/x^{2L}z)\, F_1(x^{2s}z)\, G_1(1/x^{2s}z)}{R_1(x^{2s}z)\, S_1(1/x^{2s}z)\, F_1(x^{2L}z)\, G_1(1/x^{2L}z)} \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} V_1^{(r)}(z) = & -(-1)^n (A_{n-1} b)^{(s-L)/L} \left(\frac{bx^s}{z}\right) \frac{A(x^s z)\, B(1/x^s z)}{A(x^{2s})\, B(1/x^{2s})} \\ & \times \frac{X(x^{L-2s}z)\, Y(1/x^{L-2s}z)}{X(x^L z)\, Y(1/x^L z)} \\ & \times \frac{R_1(z)\, S_1(1/z)\, F_1(x^{2(L-s)}z)\, G_1(1/x^{2(L-s)}z)}{R_1(x^{2(L-s)}z)\, S_1(1/x^{2(L-s)}z)\, F_1(z)\, G_1(1/z)} \end{aligned} \tag{4.31}$$

Consider first the result (4.30) for $V_1^{(l)}(z)$, into which we substitute the solutions (4.27) and (4.28). It is straightforward to establish, along the lines discussed in Appendix C, that the contribution of the A and B functions is the same as that for the largest eigenvalue $V_0(z)$. This leaves the result

$$\begin{aligned} \frac{V_1^{(l)}(z)}{V_0(z)} & \\ = & -(A_{n-1} b)^{(L-s)/L} \left(\frac{x^s z}{b}\right) \frac{R_1(x^{2L}z)\, S_1(1/x^{2L}z)\, X(x^{2s+L}z)\, Y(1/x^{2s+L}z)}{R_1(x^{2s})\, S_1(1/x^{2s})\, X(x^L z)\, Y(1/x^L z)} \\ & \times \prod_{n=0}^{\infty} \frac{X(x^{2n(L-s)+L}z)\, X(x^{2n(L-s)+3L}z)}{X(x^{2n(L-s)+2s+L}z)\, X(x^{2n(L-s)-2s+3L}z)} \\ & \times \frac{Y(1/x^{2n(L-s)+L}z)\, Y(1/x^{2n(L-s)+3L}z)}{Y(1/x^{2m(L-s)+2s+L}z)\, Y(1/x^{2n(L-s)-2s+3L}z)} \end{aligned}$$

$$\times \prod_{m=0}^{\infty} \frac{R_1(x^{(4m+4)s+2n(L-s)}z)\, R_1(x^{(4m+4)s+2n(L-s)+2L}z)}{R_1(x^{(4m+6)s+2n(L-s)}z)\, R_1(x^{4ms+2n(L-s)+2L}z)}$$
$$\times \frac{R_1(x^{(4m-2)s+2n(L-s)+4L}z)\, S_1(x^{(4m-2)s-2n(L-s)}/z)}{R_1(x^{4ms+2n(L-s)+4L}z)\, S_1(x^{4ms-2n(L-s)}/z)}$$
$$\times \frac{S_1(x^{(4m+4)s-2n(L-s)-2L}/z)\, S_1(x^{(4m+4)s-2n(L-s)-4L}/z)}{S_1(x^{4ms-2n(L-s)-2L}/z)\, S_1(x^{(4m+6)s-2n(L-s)-4L}/z)} \tag{4.32}$$

This expression can in turn be simplified. Again proceeding as in Appendix C, we arrive at the considerably simplified result for $z > 1$,

$$\frac{V_1^{(l)}(z)}{V_0(z)} = -(A_{n-1}b)^{(L-s)/L} \left(\frac{x^s z}{b}\right) \frac{E(a_n/z, x^{4s})\, E(a_{n+1}/z, x^{4s})}{E(x^{2s}z/a_n, x^{4s})\, E(x^{2s}z/a_{n+1}, x^{4s})} \tag{4.33}$$

On the other hand, the result obtained by simplifying the expression (4.31) for $z < 1$ is

$$\frac{V_1^{(r)}(z)}{V_0(z)} = -(A_{n-1}b)^{(s-L)/L} \left(\frac{x^s b}{z}\right) \frac{E(z/a_n, x^{4s})\, E(z/a_{n+1}, x^{4s})}{E(x^{2s}z/a_n, x^{4s})\, E(x^{2s}z/a_{n+1}, x^{4s})} \tag{4.34}$$

which is compatible with (4.33) provided

$$(A_{n-1}b)^{2(L-s)/L} = \frac{b^2}{a_n a_{n+1}} \tag{4.35}$$

However, this result can be seen to follow from (4.17) and (4.19) with $a = 0$.

The final result for the 1-string eigenvalues can be written in the form

$$\frac{V_1(w)}{V_0(w)} = \pm \frac{w}{(a_n a_{n+1})^{1/2}} \frac{E(x^s a_n/w, x^{4s})\, E(x^s a_{n+1}/w, x^{4s})}{E(x^s w/a_n, x^{4s})\, E(x^s w/a_{n+1}, x^{4s})} \tag{4.36}$$

which is explicitly dependent on the location of the holes a_n and a_{n+1}, which in turn are implicitly dependent on the location of the excitation b. In addition, we also notice that elliptic functions now appear with the new nome x^{4s}.

Up to this point, we have not made explicit use of the second Bethe ansatz equation (4.18). In the low-temperature limit the corresponding equation (4.7) gave solutions for the location b of the excitations. In the general case, substitution of the solutions (4.27) and (4.28) into (4.18) yields, after simplification, the simple result

$$(A_{n-1}b)^{2s/L} = \frac{E(x^L a_n/b, x^{2(L-s)})\, E(x^L a_{n+1}/b, x^{2(L-s)})}{E(x^L b/a_n, x^{2(L-s)})\, E(x^L b/a_{n+1}, x^{2(L-s)})} \tag{4.37}$$

Notice that the excitations b depend on the location of the two holes and that another new nome $x^{2(L-s)}$ has appeared. In a similar manner, the first Bethe ansatz equation (4.17) simplifies to

$$a^{n+1}\left[\frac{E(x^s/a,\, x^{4s})}{E(x^s a,\, x^{4s})}\right]^N + (-1)^{n+1}\, b(A_{n-1}b)^{(2s-L)/L}\, \frac{E(x^L b/a,\, x^{2L-2s})}{E(x^L a/b,\, x^{2L-2s})}$$
$$\times \prod_{j=1}^{n+1} \frac{E(x^{2s},\, x^{2L-2s})}{E(x^{2s}a_j/a,\, x^{2L-2s})}\, \mathscr{T}\left(\frac{a}{a_j},\, x^{2s},\, x^{2L-2s}\right) = 0 \tag{4.38}$$

where the function $\mathscr{T}(w, y, p)$ is defined in (3.24). Multiplying the two equations for the holes together and using (4.37) leads to the result

$$\left[\frac{1}{(a_n a_{n+1})^{1/2}}\, \frac{E(x^s a_n,\, x^{4s})\, E(x^s a_{n+1},\, x^{4s})}{E(x^s/a_n,\, x^{4s})\, E(x^s/a_{n+1},\, x^{4s})}\right]^N = 1 \tag{4.39}$$

where we have made use of the identity[2]

$$\mathscr{T}(w, y, p)\, \mathscr{T}\left(\frac{1}{w}, y, p\right) = \frac{E(yw, p)\, E(y/w, p)}{E(y, p)^2} \tag{4.40}$$

The result (4.39) ensures that the 1-string eigenvalues (4.36) are Nth roots of unity at $w = 1$, where the transfer matrix reduces to the shift operator.

4.2. 2-Strings

In this section we consider excitations for which two zeros depart from the unit circle. These zeros are excited to the circles $|z| = x^{-s}$ and $|z| = x^s$. We denote the zeros by

$$\begin{aligned} z_j &= a_j \qquad \text{for} \quad j = 1, \ldots, n-2 \\ z_{n-1} &= b_2 x^{-s} \\ z_n &= b_1 x^s \end{aligned} \tag{4.41}$$

with $|a| \sim |b_1| \sim |b_2| \sim 1$. With these substitutions, the Bethe ansatz equations (2.44) are

$$a^n \left[\frac{E(x^s/a)}{E(x^s a)}\right]^N + (-1)^n\, (A_{n-2}b_1 b_2)^{(2s-L)/L}$$
$$\times \frac{E(x^s b_2/a)\, E(x^{3s}b_1/a)}{E(x^s a/b_1)\, E(x^{3s}a/b_2)} \prod_{j=1}^{n-2} \frac{E(x^{2s}a_j/a)}{E(x^{2s}a/a_j)} = 0 \tag{4.42}$$

$$b_1^n x^{ns}\left[\frac{E(1/b_1)}{E(x^{2s}b_1)}\right]^N + (-1)^n (A_{n-2}b_1b_2)^{(2s-L)/L}$$
$$\times \frac{E(b_2/b_1)}{E(x^{4s}b_1/b_2)} \prod_{j=1}^{n-2} \frac{E(x^s a_j/b_1)}{E(x^{3s}b_1/a_j)} = 0 \tag{4.43}$$

$$b_2^n x^{-ns}\left[\frac{E(x^{2s}/b_2)}{E(b_2)}\right]^N + (-1)^n (A_{n-2}b_1b_2)^{(2s-L)/L}$$
$$\times \frac{E(x^{4s}b_1/b_2)}{E(b_2/b_1)} \prod_{j=1}^{n-2} \frac{E(x^{3s} a_j/b_2)}{E(x^s b_2/a_j)} = 0 \tag{4.44}$$

with $a \equiv a_k$, $k = 1,\dots, n-2$.

Keeping the leading terms in the low-temperature limit, $x \to 0$, we find that these equations reduce to

$$a^n + (-1)^n (A_{n-2}b_1b_2)^{(2s-L)/L} = 0 \tag{4.45}$$

$$b_1^n x^{ns}\left(1-\frac{1}{b_1}\right)^N + (-1)^n (A_{n-2}b_1b_2)^{(2s-L)/L}\left(1-\frac{b_2}{b_1}\right) = 0 \tag{4.46}$$

$$b_2^n\left(1-\frac{b_2}{b_1}\right)(1-b_2)^N + (-1)^n (A_{n-2}b_1b_2)^{(2s-L)/L} x^{ns} = 0 \tag{4.47}$$

From the last two equations we expect that $b_1 \sim b_2 \sim b$. This relation in fact holds if either $x \to 0$ or $N \to \infty$. Elimination of the term $(1 - b_2/b_1)$ in these two equations gives a single equation for b. This indicates the appropriate way to treat the general equations to obtain two equations, one for a and one for b. Eliminating the common factor $E(b_2/b_1)/E(x^{4s}b_1/b_2)$ in (4.43) and (4.44) yields

$$\left(\frac{b_2}{b_1}\right)^n = (A_{n-2}b_1b_2)^{2(2s-L)/L}\left[\frac{E(b_2)\,E(x^{2s}b_1)}{E(b_1)\,E(x^{2s}/b_2)}\right]^N$$
$$\times \prod_{j=1}^{n-2} \frac{E(x^s a_j/b_1)\,E(x^{3s}a_j/b_2)}{E(x^s b_2/a_j)\,E(x^{3s}b_1/a_j)} \tag{4.48}$$

In the low-temperature limit, $x \to 0$, this gives

$$\left(\frac{b_2}{b_1}\right)^n = (A_{n-2}b_1b_2)^{2(2s-L)/L}\left(\frac{1-b_2}{1-b_1}\right)^N \tag{4.49}$$

which, on setting $b_1 = b_2 = b$, leads to

$$(A_{n-2}b^2)^{2(2s-L)/L} = 1 \tag{4.50}$$

Combining (4.45) and (4.50), we obtain

$$a^n + (-1)^n (\pm 1) = 0 \tag{4.51}$$

where in addition

$$A_{n-2}^N = 1 \tag{4.52}$$

Setting

$$\prod_{j=1}^{n} (a - a_j) = \text{l.h.s. of } (4.51) \tag{4.53}$$

yields $A_n = (\pm 1)$, so that the two holes a_{n-1} and a_n must satisfy

$$A_{n-2} a_{n-1} a_n = \pm 1 \tag{4.54}$$

As for the 1-string case, we have to check the consistency of these equations. As an example, we return to the values $L = 3$, $s = 1$, $N = 2n = 6$ discussed in Section 4.1 for the 1-strings. From (4.51) and (4.52), we have

$$a^3 - (\pm 1) = 0 \qquad \text{with} \quad A_1 = \pm 1, e^{\pm \pi i/3}, e^{\pm 2\pi i/3} \tag{4.55}$$

The possible values of b follow from (4.50), which is simply $A_1 b^2 = \pm 1$. Finally, in this example the constraint (4.54) is simply $A_1 a_2 a_3 = \pm 1$ with $A_1 = a_1$. The 12 possible 2-string excitations are shown in Fig. 5. Each eigenvalue lies in the w band, that is, each eigenvalue satisfies

$$|V_2(z)| = w \tag{4.56}$$

Along with the 18 1-string eigenvalues, we thus have 12 2-string eigenvalues, for a total of 30 eigenvalues. This exhausts the leading w band of eigenvalues.

To carry out the perturbation argument, the functions $A(z)$, $B(1/z)$, $F(z)$, and $G(1/z)$ are defined as in (3.8) and (3.9). Because we treat the thermodynamic limit, it suffices to impose the relation $b_1 = b_2 = b$, which holds up to exponentially small corrections for large N. We can thus use the previous definitions (4.15) of the functions $X(z)$ and $Y(1/z)$. However, the functions $R_1(z)$ and $S_1(1/z)$ in (4.16) are replaced by

$$R_2(z) = \prod_{j=n-1}^{n} \prod_{k=0}^{\infty} (1 - x^{2Lk} z/z_j), \qquad S_2(1/z) = \prod_{j=n-1}^{n} \prod_{k=1}^{\infty} (1 - x^{2Lk} z_j/z) \tag{4.57}$$

With these definitions, (4.42) can be written as

$$a^n \frac{B(1/x^{2L-s}a)\, G(1/x^{2s}a)\, Y(1/x^s a)\, Y(1/x^{3s}a)\, S_2(1/x^{2(L-s)}a)}{B(1/x^s a)\, G(1/x^{2(L-s)}a)\, Y(1/x^{2L-s}a)\, Y(1/x^{2L-3s}a)\, S_2(1/x^{2s}a)}$$
$$+ (-1)^n (A_{n-2} b^2)^{(2s-L)/L}$$
$$\times \frac{A(x^s a)\, F(x^{2(L-s)}a)\, X(x^{2L-s}a)\, X(x^{2L-3s}a)\, R_2(x^{2s}a)}{A(x^{2L-s}a)\, F(x^{2s}a)\, X(x^s a)\, X(x^{3s}a)\, R_2(x^{2(L-s)}a)} = 0 \quad (4.58)$$

This time we write

$$\prod_{j=1}^{n} (a - a_j) = \text{l.h.s. of } (4.58) \quad (4.59)$$

Again equating dominant terms for $|a| < 1$ and $|a| > 1$, we obtain

$$\frac{A(x^s a)\, F(x^{2(L-s)}a)\, X(x^{2L-s}a)\, X(x^{2L-3s}a)\, R_2(x^{2s}a)}{A(x^{2L-s}a)\, F(x^{2s}a)\, X(x^s a)\, X(x^{3s}a)\, R_2(x^{2(L-s)}a)}$$
$$= \prod_{j=1}^{n} \left(1 - \frac{a}{a_j}\right) = \frac{F(a)}{F(x^{2L}a)} \quad (4.60)$$

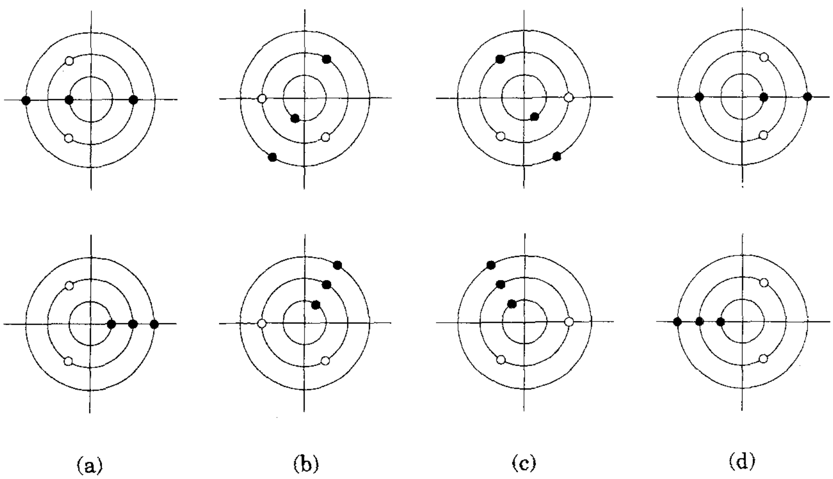

Fig. 5. Schematic representation of the 2-string Bethe ansatz zeros in the low-temperature limit for $N=6$, $L=3$, and $s=2$, with (a) $A_1=1$, (b) $A_1=\exp(\pi i/3)$, (c) $A_1=\exp(2\pi i/3)$, (d) $A_1=-1$. Here $A_1=a_1$ is the zero on the unit (middle) circle on which the holes are denoted by open circles. The inner circle is of radius $|z|=x^s$ and the outer circle is of radius $|z|=x^{-s}$. The distribution of zeros for the remaining cases, $A_1=\exp(-2\pi i/3)$ and $A_1=\exp(-\pi i/3)$, are conjugate to (c) and (b), respectively.

and

$$\frac{B(1/x^{2L-s}a)\,G(1/x^{2s}a)\,Y(1/x^{s}a)\,Y(1/x^{3s}a)\,S_2(1/x^{2(L-s)}a)}{B(1/x^{s}a)\,G(1/x^{2(L-s)}a)\,Y(1/x^{2L-s}a)\,Y(1/x^{2L-3s}a)\,S_2(1/x^{2s}a)}$$

$$=\prod_{j=1}^{n}\left(1-\frac{a_j}{a}\right)=\frac{G(1/x^{2L}a)}{G(1/a)} \tag{4.61}$$

To solve these equations, we introduce the functions

$$\tilde{F}(a)=\frac{F(a)}{F(x^{2(L-s)}a)}, \qquad \tilde{G}\left(\frac{1}{a}\right)=\frac{G(1/a)}{G(1/x^{2(L-s)}a)} \tag{4.62}$$

and write (4.60) and (4.61) in the form

$$\tilde{F}(a)=\frac{A(x^{s}a)\,X(x^{2L-s}a)\,X(x^{2L-3s}a)\,R_2(x^{2s}a)}{A(x^{2L-s}a)\,X(x^{s}a)\,X(x^{3s}a)\,R_2(x^{2(L-s)}a)}\,\frac{1}{\tilde{F}(x^{2s}a)} \tag{4.63}$$

$$\tilde{G}\left(\frac{1}{a}\right)=\frac{B(x^{s}/a)\,Y(x^{3s-2L}/a)\,Y(x^{5s-2L}/a)\,S_2(1/a)}{B(x^{3s-2L}/a)\,Y(x^{s}/a)\,Y(x^{-s}/a)\,S_2(x^{4s-2L}/a)}\,\frac{1}{\tilde{G}(1/x^{2s}a)} \tag{4.64}$$

Solving now by direct recursion yields

$$\tilde{F}(a)=\frac{X(x^{2L-3s}a)}{X(x^{s}a)}$$
$$\times\prod_{m=0}^{\infty}\frac{A(x^{(4m+1)s}a)\,A(x^{(4m+1)s+2L}a)\,R_2(x^{(4m+2)s}a)\,R_2(x^{4ms+2L}a)}{A(x^{(4m+3)s}a)\,A(x^{(4m-1)s+2L}a)\,R_2(x^{(4m+4)s}a)\,R_2(x^{(4m-2)s+2L}a)} \tag{4.65}$$

and

$$\tilde{G}\left(\frac{1}{a}\right)=\frac{Y(x^{3s-2L}/a)}{Y(x^{-s}/a)}$$
$$\times\prod_{m=0}^{\infty}\frac{B(x^{(4m+1)s}/a)\,B(x^{(4m+5)s-2L}/a)\,S_2(x^{4ms}/a)\,S_2(x^{(4m+6)s-2L}/a)}{B(x^{(4m+3)s}/a)\,B(x^{(4m+3)s-2L}/a)\,S_2(x^{(4m+2)s}/a)\,S_2(x^{(4m+4)s-2L}/a)} \tag{4.66}$$

Solving (4.62) recursively then gives the solutions

$$F(a)=\prod_{n=0}^{\infty}\tilde{F}(x^{2n(L-s)}a) \tag{4.67}$$

$$G(1/a)=\prod_{n=0}^{\infty}\tilde{G}(1/x^{2n(L-s)}a) \tag{4.68}$$

Again, in the thermodynamic limit, we write the expression (2.43) for the eigenvalue $V(z)$ in the form

$$V_2(z)=\begin{cases} V_2^{(l)}(z) & \text{for} \quad z>1 \\ V_2^{(r)}(z) & \text{for} \quad z<1 \end{cases} \tag{4.69}$$

with, e.g.,

$$V_2^{(l)}(z)= -(-1)^n\,(A_{n-2}b^2)^{(L-s)/L}\left(\frac{x^s z}{b}\right)\frac{A(x^{2L-s}z)\,B(1/x^{2L-s}z)}{A(x^{2s})\,B(1/x^{2s})}$$
$$\times\frac{F(x^{2s}z)\,G(1/x^{2s}z)\,R_2(x^{2L}z)\,S_2(1/x^{2L}z)\,X(x^{3s}z)\,Y(1/x^{3s}z)}{F(x^{2L}z)\,G(1/x^{2L}z)\,R_2(x^{2s})\,S_2(1/x^{2s})\,X(x^{2L-s}z)\,Y(1/x^{2L-s}z)} \tag{4.70}$$

As in (4.30), we can establish that the contribution from the A and B functions yields the largest eigenvalue $V_0(z)$. Thus, we obtain the result

$$\frac{V_2^{(l)}(z)}{V_0(z)}= -(A_{n-2}b^2)^{(L-s)/L}\left(\frac{x^s z}{b}\right)\frac{R_2(x^{2L}z)\,S_2(1/x^{2L}z)\,X(x^{3s}z)\,Y(1/x^{3s}z)}{R_2(x^{2s})\,S_2(1/x^{2s})\,X(x^{2L-s}z)\,Y(1/x^{2L-s}z)}$$
$$\times\prod_{n=0}^{\infty}\frac{X(x^{2n(L-s)+2L-s}z)\,X(x^{2n(L-s)+2L+s}z)\,Y(1/x^{2n(L-s)+2L-s}z)}{X(x^{2n(L-s)+3s}z)\,X(x^{2n(L-s)+4L-3s}z)\,Y(1/x^{2n(L-s)+3s}z)}$$
$$\times\frac{Y(1/x^{2n(L-s)+2L+s}z)}{Y(1/x^{2n(L-s)+4L-3s}z)}\prod_{m=0}^{\infty}\frac{R_2(x^{(4m+4)s+2n(L-s)}z)}{R_2(x^{(4m+6)s+2n(L-s)}z)}$$
$$\times\frac{R_2(x^{(4m+4)s+2n(L-s)+2L}z)\,R_2(x^{(4m-2)s+2n(L-s)+4L}z)}{R_2(x^{4ms+2n(L-s)+2L}z)\,R_2(x^{4ms+2n(L-s)+4L}z)}$$
$$\times\frac{S_2(x^{(4m-2)s-2n(L-s)}/z)}{S_2(x^{4ms-2n(L-s)}/z)}$$
$$\times\frac{S_2(x^{(4m+4)s-2n(L-s)-2L}/z)\,S_2(x^{(4m+4)s-2n(L-s)-4L}/z)}{S_2(x^{4ms-2n(L-s)-2L}/z)\,S_2(x^{(4m+6)s-2n(L-s)-4L}/z)} \tag{4.71}$$

This expression can again be dramatically simplified, to yield the result

$$\frac{V_2^{(l)}(z)}{V_0(z)}= -(A_{n-2}b^2)^{(L-s)/L}\left(\frac{x^s z}{b}\right)\frac{E(a_{n-1}/z,\,x^{4s})\,E(a_n/z,\,x^{4s})}{E(x^{2s}z/a_{n-1},\,x^{4s})\,E(x^{2s}z/a_n,\,x^{4s})} \tag{4.72}$$

In particular, we note the direct similarity with (4.33), the corresponding result for the 1-string excitations. The general result,

$$\frac{V_2(w)}{V_0(w)}=\pm\frac{w}{(a_{n-1}a_n)^{1/2}}\frac{E(x^s a_{n-1}/w,\,x^{4s})\,E(x^s a_n/w,\,x^{4s})}{E(x^s w/a_{n-1},\,x^{4s})\,E(x^s w/a_n,\,x^{4s})} \tag{4.73}$$

follows from (4.72), consideration of the expression for $V_2^{(r)}(z)$, and the result

$$(A_{n-2}b^2)^{2(L-s)/L} = \frac{b^2}{a_{n-1}a_n} \tag{4.74}$$

which follows from (4.58) and (4.59) with $a=0$.

Returning to (4.48), we find that the equation determining the location of the excitations is

$$(A_{n-2}b^2)^{2(2s-L)/L} = \frac{E(x^s a_{n-1}/b, x^{2(L-s)})\, E(x^s a_n/b, x^{2(L-s)})}{E(x^s b/a_{n-1}, x^{2(L-s)})\, E(x^s b/a_n, x^{2(L-s)})} \tag{4.75}$$

This equation, corresponding to (4.37) in the 1-string case, is again explicitly dependent on the location of the two holes. Similarly, we find that the equation for the a's is

$$a^n \left[\frac{E(x^s/a, x^{4s})}{E(x^s a, x^{4s})}\right]^N + (-1)^n\, b(A_{n-2}b^2)^{(2s-L)/L} \frac{E(x^L b/a, x^{2L-2s})}{E(x^L a/b, x^{2L-2s})} \times \prod_{j=n-1}^{n} \frac{E(x^{2s}, x^{2L-2s})}{E(x^{2s}a_j/a, x^{2L-2s})} \mathcal{T}\left(\frac{a}{a_j}, x^{2s}, x^{2L-2s}\right) = 0 \tag{4.76}$$

with the holes a_{n-1} and a_n, further satisfying

$$\left[\frac{1}{(a_{n-1}a_n)^{1/2}} \frac{E(x^s a_{n-1}, x^{4s})\, E(x^s a_n, x^{4s})}{E(x^s/a_{n-1}, x^{4s})\, E(x^s/a_n, x^{4s})}\right]^N = 1 \tag{4.77}$$

So that at $w=1$ the 2-string eigenvalues (4.73) are explicitly seen to be Nth roots of unity, as they must.

5. LEADING EXCITATIONS ($2s > L$)

We have seen in the preceding section that the first band of excitations is composed of 1-string and 2-string excitations for $2s < L$. When $2s > L$ the situation becomes more complicated, however, and longer strings occur. The classification of these excitations is analogous to that of the eight-vertex model excitations in the corresponding region.[15,16] The excitations are classified into two types, according to the values of L and s. Specifically, for r excitations off the unit circle, an r-string is classified as

Type I: $\quad 1 \leqslant r \leqslant \left[\dfrac{L}{L-s}\right] - 1$

Type II: $\quad r = \left[\dfrac{L}{L-s}\right] \quad$ or $\quad r = \left[\dfrac{L}{L-s}\right] + 1$

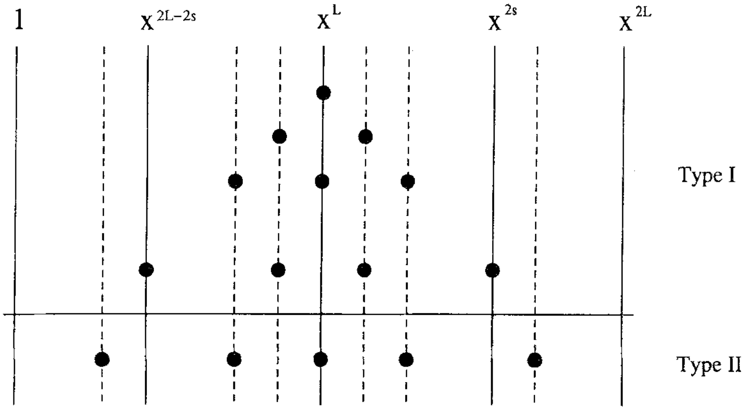

Fig. 6. Pictorial representation of string excitations off the unit circle for the case $2s > L$ showing excitations of types I and II.

where again $[\cdots]$ denotes the integer part. A schematic picture of these r-string excitations is shown within the period annulus in Fig. 6. The key circles in the complex z plane are indicated by the heavy vertical lines with the unit circle labeled by 1. Each of the vertical lines represents a possible circle of specified radius for the excitations. We see that the 1-string excitation is of type I with one zero on the circle $|z| = x^L$. The dashed lines on either side of x^L are incremented in units of $L - s$. In this way the 2-string excitations appear on the circles $|z| = x^s$ and $|z| = x^{2L-s} \equiv x^{-s}$. In general a type I excitation must satisfy $L + (r-1)(L-s) \leqslant 2s$, i.e.,

$$r(L-s) \leqslant s \tag{5.1}$$

while for each type II excitation we have

$$r(L-s) > s \tag{5.2}$$

In the rest of this section we use the Wiener–Hopf perturbation method to calculate the eigenvalues corresponding to type I and type II r-string excitations.

5.1. Type I Strings, $r(L-s) < s$

Consider a general r-string of type I with r satisfying (5.1) with strict inequality. The n zeros are denoted by

$$z_j = a_j, \qquad j = 1,..., n-r \tag{5.3}$$

with the other zeros of the form

$$z_{n-r+\mu}=b_\mu x^{L+(r+1-2\mu)(L-s)}, \qquad \mu=1,\dots,r \tag{5.4}$$

with $|a_j|\sim|b_\mu|\sim 1$. With these substitutions the Bethe ansatz equations (2.44) are

$$a^{n-r}\left[\frac{E(x^s/a)}{E(x^s a)}\right]^N+(-1)^{n-r}(A_{n-r}B_r)^{2(s-L)/L}A_{n-r}\prod_{j=1}^{n-r}\frac{E(x^{2s}a_j/a)}{E(x^{2s}a/a_j)}$$
$$\times\prod_{\mu=1}^{r}\frac{E(x^{(r+3-2\mu)s-(r+2-2\mu)L}b_{r+1-\mu}/a)}{E(x^{(r+3-2\mu)s-(r+2-2\mu)L}a/b_\mu)}=0 \tag{5.5}$$

with $a=a_k$, $k=1,\dots,n-r$, and

$$b_\mu^n x^{n(r+2-2\mu)L-n(r+1-2\mu)s}\left[\frac{E(x^{(r+2-2\mu)s-(r+2-2\mu)L}/b_\mu)}{E(x^{(r+2-2\mu)L-(r-2\mu)s}b_\mu)}\right]^N$$
$$+(-1)^n(A_{n-r}B_r)^{(2s-L)/L}x^{r(2s-L)}\prod_{\nu=1}^{r}\frac{E(x^{2(\mu-\nu)(L-s)+2s}b_\nu/b_\mu)}{E(x^{2(\nu-\mu)(L-s)+2s}b_\mu/b_\nu)}$$
$$\times\prod_{j=1}^{n-r}\frac{E(x^{(r+3-2\mu)s-(r+2-2\mu)L}a_j/b_\mu)}{E(x^{(r+2-2\mu)L-(r-1-2\mu)s}b_\mu/a_j)}=0 \tag{5.6}$$

for $\mu=1,\dots,r$. Here we have defined

$$B_m=\prod_{j=1}^{m}b_j \tag{5.7}$$

In order to discuss the r equations in (5.6), let us recall the corresponding treatment of the 2-strings in Section 4.2. In that case each of Eqs. (4.43) and (4.44) led to the result $b_1\sim b_2\sim b$ in the low-temperature limit or in the thermodynamic limit. We then eliminated a common factor from both the equations, resulting in (4.48), giving information on the values of b. For the r-string equations, we proceed in a similar manner, eliminating a common factor between consecutive pairs of equations. The net result of this process is the single equation

$$(A_{n-r}B_r)^{2r(L-s)/L}=\prod_{\mu=1}^{r}\left[b_\mu\frac{E(x^{(r+2-2\mu)(L-s)}/b_{r+1-\mu})}{E(x^{(r+2-2\mu)(L-s)}b_\mu)}\right]^N$$
$$\times\prod_{j=1}^{n-r}\prod_{\mu=1}^{r}\frac{E(x^{(r+3-2\mu)s-(r+2-2\mu)L}a_j/b_\mu)}{E(x^{(r+3-2\mu)s-(r+2-2\mu)L}b_{r+1-\mu}/a_j)} \tag{5.8}$$

In the limit $x\to 0$, (5.5) reduces to

$$a^{n-r}+(-1)^{n-r}(A_{n-r}B_r)^{2(s-L)/L}A_{n-r}=0 \tag{5.9}$$

and so, setting

$$\prod_{j=1}^{n-r} (a-a_j) = \text{l.h.s. of (5.9)} \tag{5.10}$$

yields

$$(A_{n-r}B_r)^{2(L-s)/L} = 1 \tag{5.11}$$

On the other hand, from the equations for the b_μ, (5.6), we argue that in this limit

$$b_\mu = b, \qquad \mu = 1,..., r \tag{5.12}$$

where, from (5.8) and (5.11), b satisfies

$$b^N = 1 \tag{5.13}$$

From (5.11), we also have

$$A_{n-r}b^r = \exp\left(\frac{i\pi kL}{L-s}\right) \qquad \text{for} \quad k = 0, 1,..., 2(L-s)-1 \tag{5.14}$$

Hence, given b from (5.13), this gives A_{n-r} with the $n-r$ roots a given by the solutions of (5.9). In this way we see, for example, that there are $2N(L-s)$ 1-string solutions as suggested by Fig. 3. In this limit we find that the eigenvalue expression (2.43) reduces to

$$|V_I(w)| = w \tag{5.15}$$

All of the type I eigenvalues are thus seen to be in the first band of excitations. As a specific example, let us again consider the case $L=3$, $s=2$ with $N=6$. For 1-string solutions, the requirement (5.1) is simply $2s > L$. The relevant equations are

$$a^2 + A_2 = 0 \tag{5.16}$$

with

$$A_2 b = \pm 1, \qquad b = \pm 1, e^{\pm \pi i/3}, e^{\pm 2\pi i/3} \tag{5.17}$$

These equations give rise to the 12 solutions depicted in Fig. 7. However, from Table I we see there are in fact 24 eigenvalues in the w band. For this example $2L = 3s$ and the remaining 12 eigenvalues turn out to be 2-strings, as discussed further in the next section.

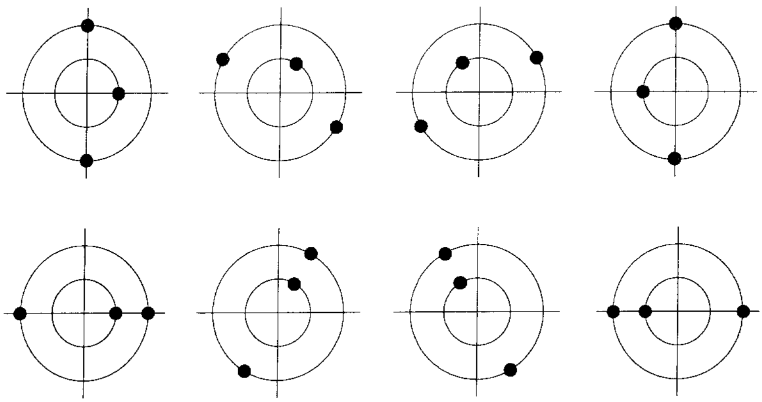

Fig. 7. Schematic representation of the type I 1-string Bethe ansatz zeros in the low-temperature limit for $N=6$, $L=3$, and $s=2$, with (a) $b=1$, (b) $b=\exp(\pi i/3)$, (c) $b=\exp(2\pi i/3)$, (d) $b=-1$. The inner and outer circles are of radii $|z|=x^L$ and $|z|=1$, respectively. The distribution of zeros for the remaining cases, $b=\exp(-2\pi i/3)$ and $b=\exp(-\pi i/3)$, are conjugate to (c) and (b), respectively.

To proceed with the perturbation argument, we define the functions $A(z)$ and $B(1/z)$ as in (3.8). However, because there are $n-r$ zeros in (5.5), the functions $F(z)$ and $G(1/z)$ are modified to

$$F_r(z)=\prod_{j=1}^{n-r}\prod_{k=0}^{\infty}(1-x^{2Lk}z/z_j), \qquad G_r(1/z)=\prod_{j=1}^{n-r}\prod_{k=1}^{\infty}(1-x^{2Lk}z_j/z) \tag{5.18}$$

As for the 2-strings in Section 4.2, it suffices in the thermodynamic limit to impose (5.12) and thus to use the previous definitions (4.15) for the functions $X(z)$ and $Y(1/z)$. This simplification leads to cancellation of a number of terms in (5.5), which we write in the form

$$\begin{aligned} a^{n-r}&\frac{B(1/x^{2L-s}a)\,G_r(1/x^{2s}a)\,Y(1/x^{(r+1)s-rL}a)\,Y(1/x^{(r-1)s-(r-2)L}a)}{B(1/x^{s}a)\,G_r(1/x^{2(L-s)}a)\,Y(1/x^{(r+2)L-(r+1)s}a)\,Y(1/x^{rL-(r-1)s}a)}\\ &+(-1)^{n-r}(A_{n-r}b^r)^{2(s-L)/L}A_{n-r}\frac{A(x^s a)\,F_r(x^{2(L-s)}a)}{A(x^{2L-s}a)\,F_r(x^{2s}a)}\\ &\times\frac{X(x^{(r+2)L-(r+1)s}a)\,X(x^{rL-(r-1)s}a)}{X(x^{(r+1)s-rL}a)\,X(x^{(r-1)s-(r-2)L}a)}=0 \end{aligned} \tag{5.19}$$

In a similar manner, we write (5.8) as

$$(A_{n-r}b^r)^{2r(L-s)/L} = b^N \frac{A(x^{2L-r(L-s)}b)\,B(1/x^{2L-r(L-s)}b)}{A(x^{r(L-s)}b)\,B(1/x^{r(L-s)}b)}$$

$$\times \frac{F_r(x^{s+r(L-s)}b)\,F_r(x^{2L-s+r(L-s)}b)}{F_r(x^{s-r(L-s)}b)\,F_r(x^{2L-s-r(L-s)}b)}$$

$$\times \frac{G_r(1/x^{s+r(L-s)}b)\,G_r(1/x^{2L-s+r(L-s)}b)}{G_r(1/x^{s-r(L-s)}b)\,G_r(1/x^{2L-s-r(L-s)}b)} \qquad (5.20)$$

To treat the first equation, we write

$$\prod_{j=1}^{n-r} (a-a_j) = \text{l.h.s. of } (5.19) \qquad (5.21)$$

Again equating dominant terms for $|a|<1$ and $|a|>1$ yields

$$\frac{A(x^s a)\,F_r(x^{2(L-s)}a)\,X(x^{(r+2)L-(r+1)s}a)\,X(x^{rL-(r-1)s}a)}{A(x^{2L-s}a)\,F_r(x^{2s}a)\,X(x^{(r+1)s-rL}a)\,X(x^{(r-1)s-(r-2)L}a)}$$

$$= \prod_{j=1}^{n-r}\left(1-\frac{a}{a_j}\right) = \frac{F_r(a)}{F_r(x^{2L}a)} \qquad (5.22)$$

and

$$\frac{B(1/x^{2L-s}a)\,G_r(1/x^{2s}a)\,Y(1/x^{(r+1)s-rL}a)\,Y(1/x^{(r-1)s-(r-2)L}a)}{B(1/x^s a)\,G_r(1/x^{2(L-s)}a)\,Y(1/x^{(r+2)L-(r+1)s}a)\,Y(1/x^{rL-(r-1)s}a)}$$

$$= \prod_{j=1}^{n-r}\left(1-\frac{a_j}{a}\right) = \frac{G_r(1/x^{2L}a)}{G_r(1/a)} \qquad (5.23)$$

Proceeding as before, we solve these equations by recursion, with the result

$$F_r(a) = \prod_{n=0}^{\infty} \tilde{F}_r(x^{2n(L-s)}a) \qquad (5.24)$$

$$G_r(1/a) = \prod_{n=0}^{\infty} \tilde{G}_r(1/x^{2n(L-s)}a) \qquad (5.25)$$

where

$$\tilde{F}_r(a)=\prod_{m=0}^{\infty}\frac{A(x^{(4m+1)s}a)\,A(x^{(4m+1)s+2L}a)\,X(x^{(4m+1)s+r(L-s)}a)}{A(x^{(4m+3)s}a)\,A(x^{(4m-1)s+2L}a)\,X(x^{(4m+1)s-r(L-s)}a)}$$
$$\times\frac{X(x^{(4m+3)s-r(L-s)}a)\,X(x^{(4m-1)s+r(L-s)+2L}a)\,X(x^{(4m+1)s-r(L-s)+2L}a)}{X(x^{(4m+3)s+r(L-s)}a)\,X(x^{(4m-1)s-r(L-s)+2L}a)\,X(x^{(4m+1)s+r(L-s)+2L}a)} \tag{5.26}$$

and

$$\tilde{G}_r\left(\frac{1}{a}\right)=\prod_{m=0}^{\infty}\frac{B(x^{(4m+1)s}/a)\,B(x^{(4m+5)s-2L}/a)\,Y(x^{(4m+1)s-r(L-s)}/a)}{B(x^{(4m+3)s}/a)\,B(x^{(4m+3)s-2L}/a)\,Y(x^{(4m+1)s+r(L-s)}/a)}$$
$$\times\frac{Y(x^{(4m+3)s+r(L-s)}/a)\,Y(x^{(4m+3)s-r(L-s)-2L}/a)\,Y(x^{(4m+5)s+r(L-s)-2L}/a)}{Y(x^{(4m+3)s-r(L-s)}/a)\,Y(x^{(4m+3)s+r(L-s)-2L}/a)\,Y(x^{(4m+5)s-r(L-s)-2L}/a)} \tag{5.27}$$

In the thermodynamic limit, we can again write the eigenvalue expression (2.43) for $V(z)$ in the form

$$V_l(z)=\begin{cases}V_l^{(l)}(z) & \text{for} \quad z>1\\ V_l^{(r)}(z) & \text{for} \quad z<1\end{cases} \tag{5.28}$$

with

$$V_l^{(l)}(z)=-(-1)^n\,(A_{n-r}b^r)^{(L-s)/L}\,x^{r(L-s)}\,\frac{A(x^{2L-s}z)\,B(1/x^{2L-s}z)}{A(x^{2s}z)\,B(1/x^{2s}z)}$$
$$\times\frac{X(x^{(r+1)s-rL}z)\,Y(x^{rL-(r+1)s}/z)\,F_r(x^{2s}z)\,G_r(1/x^{2s}z)}{X(x^{rL-(r-1)s}z)\,Y(x^{(r-1)s-rL}/z)\,F_r(x^{2L}z)\,G_r(1/x^{2L}z)} \tag{5.29}$$

and

$$V_l^{(r)}(z)=-(-1)^n\,(A_{n-r}b^r)^{(s-L)/L}\,x^{r(L-s)}\,\frac{A(x^{s}z)\,B(1/x^{s}z)}{A(x^{2s}z)\,B(1/x^{2s}z)}$$
$$\times\frac{X(x^{(r+2)L-(r+1)s}z)\,Y(x^{(r+1)s-(r+2)L}/z)\,F_r(x^{2(L-s)}z)\,G_r(1/x^{2(L-s)}z)}{X(x^{(r-1)s-(r-2)L}z)\,Y(x^{(r-2)L-(r-1)s}/z)\,F_r(z)\,G_r(1/z)} \tag{5.30}$$

Let us consider the expression (5.29) for $V_I^{(l)}(z)$. On substitution of the solutions (5.24) and (5.25), we again find that the contribution from the A and B functions gives the largest eigenvalue $V_0(z)$. Thus we obtain

$$\frac{V_I^{(l)}(z)}{V_0(z)} = -(A_{n-r}b^r)^{(L-s)/L}\, x^{r(L-s)}\, \frac{X(x^{(r+1)s-rL}z)\; Y(x^{rL-(r+1)s}/z)}{X(x^{rL-(r-1)s}z)\; Y(x^{(r-1)s-rL}/z)}$$
$$\times \prod_{m=0}^{\infty}\prod_{n=0}^{\infty} \frac{X(x^{(4m+3)s+(2n+r)(L-s)}z)\; X(x^{(4m+5)s+(2n-r)(L-s)}z)}{X(x^{(4m+3)s+(2n-r)(L-s)}z)\; X(x^{(4m+5)s+(2n+r)(L-s)}z)}$$
$$\times \frac{X(x^{(4m-1)s+(2n-r)(L-s)+4L}z)\; X(x^{(4m+1)s+(2n+r)(L-s)+4L}z)}{X(x^{(4m-1)s+(2n+r)(L-s)+4L}z)\; X(x^{(4m+1)s+(2n-r)(L-s)+4L}z)}$$
$$\times \frac{Y(x^{(4m-1)s-(2n+r)(L-s)}/z)\; Y(x^{(4m+1)s-(2n+r)(L-s)}/z)}{Y(x^{(4m-1)s-(2n-r)(L-s)}/z)\; Y(x^{(4m+1)s-(2n+r)(L-s)}/z)}$$
$$\times \frac{Y(x^{(4m+3)s-(2n-r)(L-s)-4L}/z)\; Y(x^{(4m+5)s-(2n+r)(L-s)-4L}/z)}{Y(x^{(4m+3)s-(2n+r)(L-s)-4L}/z)\; Y(x^{(4m+5)s-(2n-r)(L-s)-4L}/z)} \tag{5.31}$$

However, after simplification, this expression is in turn equivalent to

$$\frac{V_I^{(l)}(z)}{V_0(z)} = -(A_{n-r}b^r)^{(L-s)/L}\, x^{r(L-s)} \times \frac{E(x^{(r+1)s-rL}z/b,\, x^{4s})\; E(x^{(r+1)s-rL}b/z,\, x^{4s})}{E(x^{rL-(r-1)s}z/b,\, x^{4s})\; E(x^{rL-(r-1)s}b/z,\, x^{4s})} \tag{5.32}$$

A similar treatment of (5.30) for $V_I^{(r)}(z)$ gives the same result, with, however, the equivalent prefactor $(A_{n-r}b^r)^{(s-L)/L}$. Using

$$(A_{n-r}b^r)^{2(L-s)/L} = 1 \tag{5.33}$$

which follows from (5.19) and (5.21) with $a=0$, we then have the general result

$$V_I(w)/V_0(w) = \pm\phi_r(b/w) \tag{5.34}$$

where we have defined the function

$$\phi_r(w) = \frac{1}{w}\, \frac{E(x^{r(L-s)}w,\, x^{4s})\; E(x^{r(L-s)+2s}/w,\, x^{4s})}{E(x^{r(L-s)}/w,\, x^{4s})\; E(x^{r(L-s)+2s}w,\, x^{4s})} \tag{5.35}$$

Returning to (5.20), we obtain an equation for b by using (5.24)–(5.27). After proving an identity and using (5.33), it can be written as

$$\phi_r^N(b) = 1 \tag{5.36}$$

Equation (5.36) is a higher-level Bethe ansatz equation, of the same form as those found in the hard-hexagon and hard-square models.[18,19]

5.2. Type I Strings, $r(L-s)=s$

In this section we consider values of r, L, and s satisfying the equality

$$rL=(r+1)s \tag{5.37}$$

For this case, the type I Bethe ansatz equations (5.5) can be written as

$$a^{n-r}\left[\frac{E(x^s/a)}{E(x^s a)}\right]^N+(-1)^{n-r}(A_{n-r}B_r)^{2(s-L)/L}A_{n-r}\prod_{j=1}^{n-r}\frac{E(x^{2s}a_j/a)}{E(x^{2s}a/a_j)}$$
$$\times\frac{E(b_r/a)}{E(a/b_1)}\prod_{\mu=2}^{r}\frac{E(x^{2(\mu-1)(L-s)}b_{r+1-\mu}/a)}{E(x^{2(\mu-1)(L-s)}a/b_\mu)}=0 \tag{5.38}$$

that is, we simply pull out the $\mu=1$ term in the last product of (5.5). Note that we have also used (5.37) to simplify the exponent of x. In a similar manner, the single equation corresponding to (5.8) is

$$(A_{n-r}B_r)^{2s/L}=\prod_{\mu=1}^{r}\left[b_\mu\frac{E(x^{s+2(1-\mu)(L-s)}/b_{r+1-\mu})}{E(x^{s+2(1-\mu)(L-s)}b_\mu)}\right]^N$$
$$\times\prod_{j=1}^{n-r}\left[\frac{E(a_j/b_1)}{E(b_r/a_j)}\prod_{\mu=2}^{r}\frac{E(x^{2(\mu-1)(L-s)}a_j/b_\mu)}{E(x^{2(\mu-1)(L-s)}b_{r+1-\mu}/a_j)}\right] \tag{5.39}$$

In the limit $x\to 0$, the Bethe ansatz equations (5.38) reduce to

$$a^{n-r}+(-1)^{n-r}(A_{n-r}B_r)^{2(s-L)/L}A_{n-r}\frac{1-b_r/a}{1-a/b_1}=0 \tag{5.40}$$

From the equations (5.6) for b_μ, we again have

$$b_\mu=b, \qquad \mu=1,\ldots,r \tag{5.41}$$

in this limit. Using this result, (5.39) reduces to

$$(-1)^{n-r}b^n(A_{n-r}b^r)^{(L-2s)/L}=1 \tag{5.42}$$

We can also rewrite (5.40) as

$$a^{n-r+1}+(-1)^{n-r+1}A_{n-r}b(A_{n-r}b^r)^{2(s-L)/L}=0 \tag{5.43}$$

so that we have effectively increased the order of the equation by one [cf. Eq. (5.9)]. For brevity we will label the hole by $\tilde{a}_1=a_{n-r+1}$. Setting

$$\prod_{j=1}^{n-r+1}(a-a_j)=\text{l.h.s. of (5.43)} \tag{5.44}$$

we find

$$(A_{n-r}b^r)^{2(s-L)/L} = \frac{\tilde{a}_1}{b} \tag{5.45}$$

Inserting this result in (5.42) yields

$$A_{n-r+1} = (-1)^{n-r}\, b^{n-r+1} \tag{5.46}$$

From (5.43) and (5.45) we also have

$$A_{n-r} = (-1)^{n-r}\, \tilde{a}_1^{n-r} \tag{5.47}$$

which, when substituted into (5.46), yields

$$(b/\tilde{a}_1)^{n-r+1} = 1 \tag{5.48}$$

As a specific example, let us return to the case $L = 3$, $s = 2$ with $N = 6$ and consider the 2-string solutions. From (5.43), (5.45), and (5.48) the relevant equations are

$$a^2 + a_1 \tilde{a}_1 = 0, \qquad b = \pm \tilde{a}_1 \tag{5.49}$$

We first note that (5.49) is consistent with $a_1 = -\tilde{a}_1$, which is (5.47). We also need to use (5.45), which is $a_1^2 \tilde{a}_1^3 b = 1$. These equations give rise to 12 solutions, as depicted in Fig. 8. Along with the 12 type I 1-strings discussed in Section 5.1, we then have all of the 24 eigenvalues in the w band.

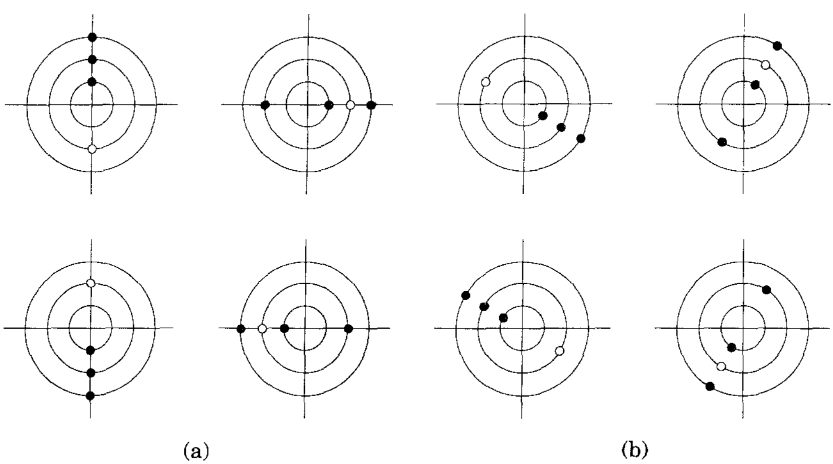

Fig. 8. Schematic representation of the type I 2-string Bethe ansatz zeros in the low-temperature limit for $N = 6$, $L = 3$, and $s = 2$, with (a) $A_2 b = 1$, (b) $A_2 b = \exp(2\pi i/3)$. The open circle indicates the position of the hole on the unit circle. The remaining circles are of radii $|z| = x^s$ and $|z| = x^{-s}$. Another four sets of zeros are conjugate to (b) with $A_2 b = \exp(-2\pi i/3)$.

To carry out the perturbation argument, we need to introduce functions $F_{r-1}(z)$ and $G_{r-1}(1/z)$ to deal with the hole $\tilde{a}_1$. Here $F_r(z)$ and $G_r(1/z)$ are defined as in (5.18). We also define the functions

$$R_m(z)=\prod_{j=1}^{m}\prod_{k=0}^{\infty}\left(1-x^{2Lk}\frac{z}{\tilde{a}_j}\right),\qquad S_m\left(\frac{1}{z}\right)=\prod_{j=1}^{m}\prod_{k=0}^{\infty}\left(1-x^{2Lk}\frac{\tilde{a}_j}{z}\right) \tag{5.50}$$

which are analogous to those defined in Sections 4.1 and 4.2. We again impose (5.41) and use the previous definitions (4.15) for the functions $X(z)$ and $Y(1/z)$. With these considerations, (5.38) and (5.39) can be written as

$$\begin{aligned}
&a^{n-r+1}\frac{B(x^{s-2L}/a)\,Y(x^{2(s-L)}/a)\,S_1(x^{2(s-L)}/a)\,G_{r-1}(1/x^{2s}a)}{B(1/x^sa)\,Y(1/x^{2s}a)\,S_1(1/x^{2s}a)\,G_{r-1}(x^{2(s-L)}/a)}\\
&\quad+(-1)^{n-r+1}A_{n-r}b(A_{n-r}b^r)^{2(s-L)/L}\frac{A(x^sa)\,X(x^{2s}a)}{A(x^{2L-s}a)\,X(x^{2(L-s)}a)}\\
&\quad\times\frac{R_1(x^{2s}a)\,F_{r-1}(x^{2(L-s)}a)}{R_1(x^{2(L-s)}a)\,F_{r-1}(x^{2s}a)}=0
\end{aligned} \tag{5.51}$$

and

$$\begin{aligned}
1&=(-1)^{n-r}b^n(A_{n-r}b^r)^{(L-2s)/L}\frac{A(x^{2L-s}b)\,B(1/x^{2L-s}b)}{A(x^sb)\,B(1/x^sb)}\\
&\quad\times\frac{R_1(x^{2(L-s)}b)\,S_1(1/x^{2(L-s)}b)\,F_{r-1}(x^{2s}b)\,G_{r-1}(1/x^{2s}b)}{R_1(x^{2s}b)\,S_1(1/x^{2s}b)\,F_{r-1}(x^{2(L-s)}b)\,G_{r-1}(1/x^{2(L-s)}b)}
\end{aligned} \tag{5.52}$$

As before, we solve (5.51) for the functions $F_{r-1}(z)$ and $G_{r-1}(1/z)$ to obtain

$$F_{r-1}(a)=\prod_{n=0}^{\infty}\tilde{F}_{r-1}(x^{2n(L-s)}a) \tag{5.53}$$

$$G_{r-1}(1/a)=\prod_{n=0}^{\infty}\tilde{G}_{r-1}(1/x^{2n(L-s)}a) \tag{5.54}$$

where

$$\begin{aligned}
\tilde{F}_{r-1}(a)=\prod_{m=0}^{\infty}&\frac{A(x^{(4m+1)s}a)\,A(x^{(4m+1)s+2L}a)\,X(x^{(4m+2)s}a)}{A(x^{(4m+3)s}a)\,A(x^{(4m-1)s+2L}a)\,X(x^{(4m+4)s}a)}\\
&\times\frac{X(x^{4ms+2L}a)\,R_1(x^{(4m+2)s}a)\,R_1(x^{4ms+2L}a)}{X(x^{(4m-2)s+2L}a)\,R_1(x^{(4m+4)s}a)\,R_1(x^{(4m-2)s+2L}a)}
\end{aligned} \tag{5.55}$$

and

$$\tilde{G}_{r-1}\left(\frac{1}{a}\right)=\prod_{m=0}^{\infty}\frac{B(x^{(4m+1)s}/a)\,B(x^{(4m+5)s-2L}/a)\,Y(x^{4ms}/a)}{B(x^{(4m+3)s}/a)\,B(x^{(4m+3)s-2L}/a)\,Y(x^{(4m+2)s}/a)}$$
$$\times\frac{Y(x^{(4m+6)s-2L}/a)\,S_1(x^{4ms}/a)\,S_1(x^{(4m+6)s-2L}/a)}{Y(x^{(4m+4)s-2L}/a)\,S_1(x^{(4m+2)s}/a)\,S_1(x^{(4m+4)s-2L}/a)} \qquad (5.56)$$

In the thermodynamic limit, the eigenvalue expression (2.43) for $V(z)$ is of the form

$$V_{le}(z)=\begin{cases}V_{le}^{(l)}(z) & \text{for}\quad z>1\\ V_{le}^{(r)}(z) & \text{for}\quad z<1\end{cases} \qquad (5.57)$$

where

$$V_{le}^{(l)}(z)=(-1)^n\,(A_{n-r}b^r)^{(L-s)/L}\,x^s\,\frac{A(x^{2L-s}z)\,B(1/x^{2L-s}z)\,X(z)}{A(x^{2s})\,B(1/x^{2s})\,X(x^{2s}z)}$$
$$\times\frac{Y(1/z)}{Y(1/x^{2s}z)}\,\frac{R_1(x^{2L}z)\,S_1(1/x^{2L}z)\,F_{r-1}(x^{2s}z)\,G_{r-1}(1/x^{2s}z)}{R_1(x^{2s}z)\,S_1(1/x^{2s}z)\,F_{r-1}(x^{2L}z)\,G_{r-1}(1/x^{2L}z)} \qquad (5.58)$$

and

$$V_{le}^{(r)}(z)=(-1)^n\,(A_{n-r}b^r)^{(s-L)/L}\,x^s\,\frac{A(x^sz)\,B(1/x^sz)\,X(x^{2L}z)}{A(x^{2s})\,B(1/x^{2s})\,X(x^{2(L-s)}z)}$$
$$\times\frac{Y(1/x^{2L}z)\,R_1(z)\,S_1(1/z)\,F_{r-1}(x^{2(L-s)}z)\,G_{r-1}(x^{2(s-L)}/z)}{Y(x^{2(s-L)}/z)\,R_1(x^{2(L-s)}z)\,S_1(x^{2(s-L)}/z)\,F_{r-1}(z)\,G_{r-1}(1/z)} \qquad (5.59)$$

Substituting the solutions (5.53) and (5.54) into the expression (5.58) for $V_{le}^{(l)}(z)$, we see that the contribution from the A and B functions is the same as that in Section 5.1, and so gives the largest eigenvalue $V_0(z)$. Thus we arrive at the somewhat unwieldy intermediate result

$$\frac{V_{le}^{(l)}(z)}{V_0(z)}$$
$$=(A_{n-r}b^r)^{(L-s)/L}\,x^s\,\frac{X(z)\,Y(1/z)}{X(x^{2s}z)\,Y(1/x^{2s}z)}$$
$$\times\prod_{m=0}^{\infty}\prod_{n=0}^{\infty}\frac{X(x^{(4m+4)s+2n(L-s)}z)\,X(x^{(4m+4)s+2n(L-s)+2L}z)}{X(x^{(4m+6)s+2n(L-s)}z)\,X(x^{4ms+2n(L-s)+2L}z)}$$
$$\times\frac{X(x^{(4m-2)s+2n(L-s)+4L}z)\,R_1(x^{(4m+4)s+2n(L-s)}z)}{X(x^{4ms+2n(L-s)+4L}z)\,R_1(x^{(4m+6)s+2n(L-s)}z)}$$

$$\times \frac{R_1(x^{(4m+4)s+2n(L-s)+2L}z)\,R_1(x^{(4m-2)s+2n(L-s)+4L}z)}{R_1(x^{4ms+2n(L-s)+2L}z)\,R_1(x^{4ms+2n(L-s)+4L}z)}$$

$$\times \frac{Y(x^{(4m-2)s-2n(L-s)}/z)\,Y(x^{(4m+4)s-2n(L-s)-2L}/z)}{Y(x^{4ms-2n(L-s)}/z)\,Y(x^{4ms-2n(L-s)-2L}/z)}$$

$$\times \frac{Y(x^{(4m+4)s-2n(L-s)-4L}/z)\,S_1(x^{(4m-2)s-2n(L-s)}/z)}{Y(x^{(4m+6)s-2n(L-s)-4L}/z)\,S_1(x^{4ms-2n(L-s)}/z)}$$

$$\times \frac{S_1(x^{(4m+4)s-2n(L-s)-2L}/z)\,S_1(x^{(4m+4)s-2n(L-s)-4L}/z)}{S_1(x^{4ms-2n(L-s)-2L}/z)\,S_1(x^{(4m+6)s-2n(L-s)-4L}/z)} \tag{5.60}$$

After the application of a suitable identity, this expression dramatically simplifies to

$$\frac{V_{Ie}^{(l)}(z)}{V_0(z)} = -(A_{n-r}b^r)^{(L-s)/L}\left(\frac{x^s z}{b}\right)\frac{E(b/z, x^{4s})\,E(\tilde{a}_1/z, x^{4s})}{E(x^{2s}z/b, x^{4s})\,E(x^{2s}z/\tilde{a}_1, x^{4s})} \tag{5.61}$$

A similar treatment of (5.59) for $V_{Ie}^{(r)}(z)$ leads to the final result

$$\frac{V_{Ie}(w)}{V_0(w)} = \pm\frac{w}{(b\tilde{a}_1)^{1/2}}\frac{E(x^s b/w, x^{4s})\,E(x^s\tilde{a}_1/w, x^{4s})}{E(x^s w/b, x^{4s})\,E(x^s w/\tilde{a}_1, x^{4s})} \tag{5.62}$$

subject to (5.45), which also holds away from the low-temperature limit.

Returning to (5.52), we find that it can be written as

$$(A_{n-r}b^r)^{(2s-L)/L} = (-1)^{n-r}\left[\sqrt{b}\,\frac{E(x^s/b, x^{4s})}{E(x^s b, x^{4s})}\right]^N$$

$$\times \frac{E(x^{2s}, x^{2L-2s})}{E(x^{2s}b/\tilde{a}_1, x^{2L-2s})}\,\mathscr{T}\left(\frac{b}{\tilde{a}_1}, x^{2s}, x^{2L-2s}\right) \tag{5.63}$$

where the function $\mathscr{T}(w, y, p)$ is defined in (3.24). On the other hand, the equation (5.51) for the a's is

$$a^{n-r+1}\left[\frac{E(x^s/a, x^{4s})}{E(x^s a, x^{4s})}\right]^N + (-1)^{n-r+1}\,(A_{n-r}b^r)^{2(s-L)/L}$$

$$\times A_{n-r}b\,\frac{E(x^{2s}, x^{2L-2s})}{E(x^{2s}b/a, x^{2L-2s})}\,\frac{E(x^{2s}, x^{2L-2s})}{E(x^{2s}\tilde{a}_1/a, x^{2L-2s})}$$

$$\times \mathscr{T}\left(\frac{a}{b}, x^{2s}, x^{2L-2s}\right)\mathscr{T}\left(\frac{a}{\tilde{a}_1}, x^{2s}, x^{2L-2s}\right) = 0 \tag{5.64}$$

Using (5.45), the hole $\tilde{a}_1$ then satisfies

$$\left[\tilde{a}_1 \frac{E(x^s/\tilde{a}_1, x^{4s})}{E(x^s\tilde{a}_1, x^{4s})}\right]^N + (-1)^{n-r+1} (A_{n-r}b^r)^{(L-2s)/L} \times \frac{E(x^{2s}, x^{2L-2s})}{E(x^{2s}b/\tilde{a}_1, x^{2L-2s})} \mathscr{T}\left(\frac{\tilde{a}_1}{b}, x^{2s}, x^{2L-2s}\right) = 0 \qquad (5.65)$$

Elimination of the common factor between this equation and (5.63) then yields

$$\left[\frac{1}{(b\tilde{a}_1)^{1/2}} \frac{E(x^s b, x^{4s})\, E(x^s\tilde{a}_1, x^{4s})}{E(x^s/b, x^{4s})\, E(x^s/\tilde{a}_1, x^{4s})}\right]^N = 1 \qquad (5.66)$$

again ensuring that the eigenvalues (5.62) are Nth roots of unity at $w=1$.

5.3. Type II Strings, $r(L-s)>s$

The last case to be considered is that of type II strings with

$$r(L-s)>s \qquad (5.67)$$

We write the Bethe ansatz equation for a corresponding to (5.5) and (5.38) in the form

$$a^{n-r+2}\left[\frac{E(x^s/a)}{E(x^s a)}\right]^N + (-1)^{n-r} (A_{n-r}B_r)^{(2s-L)/L} \frac{b_1 b_r}{B_r} \prod_{j=1}^{n-r} \frac{E(x^{2s}a_j/a)}{E(x^{2s}a/a_j)} \times \frac{E(x^{r(L-s)-s}a/b_r)}{E(x^{r(L-s)-s}b_1/a)} \prod_{\mu=1}^{r-1} \frac{E(x^{s+(r-2\mu)(L-s)}b_\mu/a)}{E(x^{s+(r-2\mu)(L-s)}a/b_{r+1-\mu})} = 0 \qquad (5.68)$$

The single equation corresponding to (5.8) and (5.39) is

$$(A_{n-r}B_r)^{2r(L-s)/L} = \frac{A_{n-r}^2}{(b_1 b_r)^{n-r}} \prod_{\mu=1}^{r} \left[b_\mu \frac{E(x^{(r+2-2\mu)(L-s)}/b_{r+1-\mu})}{E(x^{(r+2-2\mu)(L-s)}b_\mu)}\right]^N \times \prod_{j=1}^{n-r} \left[\frac{E(x^{rL-(r+1)s}b_1/a_j)}{E(x^{rL-(r+1)s}a_j/b_r)} \prod_{\mu=1}^{r-1} \frac{E(x^{s+(r-2\mu)(L-s)}a_j/b_{r+1-\mu})}{E(x^{s+(r-2\mu)(L-s)}b_\mu/a_j)}\right] \qquad (5.69)$$

In the limit $x \to 0$, these equations reduce to

$$a^{n-r+2} + (-1)^{n-r} b^{2-r}(A_{n-r}b^r)^{(2s-L)/L} = 0 \qquad (5.70)$$

and

$$(A_{n-r}b^r)^{2r(L-s)/L} = (A_{n-r}b^r)^2 \qquad (5.71)$$

where we have again set $b_\mu = b$. In this case we see that there are effectively two holes. For brevity we label them by

$$\tilde{a}_m = a_{n-r+m} \tag{5.72}$$

with now $m = 1$ and 2. This time, setting

$$\prod_{j=1}^{n-r+2} (a - a_j) = \text{l.h.s. of (5.70)} \tag{5.73}$$

gives

$$(A_{n-r} b^r)^{2(L-s)/L} = \frac{b^2}{\tilde{a}_1 \tilde{a}_2} \tag{5.74}$$

Substituting this result in (5.71) yields

$$A_{n-r}^2 = (\tilde{a}_1 \tilde{a}_2)^{-r} \tag{5.75}$$

However, from (5.70) we also have

$$A_{n-r}^2 = (\tilde{a}_1 \tilde{a}_2)^{n-r} \tag{5.76}$$

so that

$$(\tilde{a}_1 \tilde{a}_2)^n = 1 \tag{5.77}$$

These equations are again consistent, yielding eigenvalues satisfying

$$|V_{II}(w)| = w \tag{5.78}$$

As an example, for the case $L = 5$, $s = 3$ with $N = 10$, we find there are 40 type I 1-string excitations, 40 type II 2-strings, and 30 type II 3-strings, which exhausts all 110 eigenvalues in the w band.

For the perturbation argument, we work with the functions $F_{r-2}(z)$, $G_{r-2}(1/z)$, $R_2(z)$, and $S_2(1/z)$ to deal with the two holes. The functions $F_r(z)$, $G_r(1/z)$, $R_m(z)$, and $S_m(1/z)$ are defined as in (5.18) and (5.50). We will again impose the relation $b_\mu = b$ and use the previous definitions (4.15) for the functions $X(z)$ and $Y(1/z)$. With these considerations, we rewrite (5.68) and (5.69) as

$$\begin{aligned} &a^{n-r+2} \frac{B(1/x^{2L-s}a)}{B(1/x^s a)} \frac{Y(x^{r(L-s)-2L-s}/a)\, Y(x^{r(L-s)-2L+s}/a)}{Y(1/x^{r(L-s)-s}a)\, Y(1/x^{r(L-s)+s}a)} \\ &\times \frac{S_2(1/x^{2(L-s)}a)\, G_{r-2}(1/x^{2s}a)}{S_2(1/x^{2s}a)\, G_{r-2}(1/x^{2(L-s)}a)} + (-1)^{n-r} b^{2-r} (A_{n-r} b^r)^{(2s-L)/L} \\ &\times \frac{A(x^s a)}{A(x^{2L-s}a)} \frac{X(x^{r(L-s)-s}a)\, X(x^{r(L-s)+s}a)}{X(x^{2L-s-r(L-s)}a)\, X(x^{2L+s-r(L-s)}a)} \\ &\times \frac{R_2(x^{2s}a)\, F_{r-2}(x^{2(L-s)}a)}{R_2(x^{2(L-s)}a)\, F_{r-2}(x^{2s}a)} = 0 \end{aligned} \tag{5.79}$$

and

$$
\begin{aligned}
&(A_{n-r}b^r)^{2r(L-s)/L} \\
&\quad = (A_{n-r}b^r)^2 \frac{A(x^{2L-r(L-s)}b)\, B(x^{r(L-s)-2L}/b)}{A(x^{r(L-s)}b)\, B(1/x^{r(L-s)}b)} \\
&\qquad \times \frac{R_2(x^{2L-s-r(L-s)}b)\, R_2(x^{2L+s-r(L-s)}b)}{R_2(x^{r(L-s)-s}b)\, R_2(x^{r(L-s)+s}b)} \\
&\qquad \times \frac{S_2(x^{r(L-s)-2L+s}/b)\, S_2(x^{r(L-s)-2L-s}/b)}{S_2(1/x^{r(L-s)-s}b)\, S_2(1/x^{r(L-s)+s}b)} \\
&\qquad \times \frac{F_{r-2}(x^{r(L-s)-s}b)\, F_{r-2}(x^{r(L-s)+s}b)}{F_{r-2}(x^{2L-s-r(L-s)}b)\, F_{r-2}(x^{2L+s-r(L-s)}b)} \\
&\qquad \times \frac{G_{r-2}(1/x^{r(L-s)-s}b)\, G_{r-2}(1/x^{r(L-s)+s}b)}{G_{r-2}(x^{r(L-s)-2L+s}/b)\, G_{r-2}(x^{r(L-s)-2L-s}/b)}
\end{aligned}
\tag{5.80}
$$

From (5.79) we obtain

$$
\begin{aligned}
&\frac{A(x^s a)}{A(x^{2L-s}a)} \frac{X(x^{r(L-s)-s}a)\, X(x^{r(L-s)+s}a)}{X(x^{2L-s-r(L-s)}a)\, X(x^{2L+s-r(L-s)}a)} \\
&\qquad \times \frac{R_2(x^{2s}a)\, F_{r-2}(x^{2(L-s)}a)}{R_2(x^{2(L-s)}a)\, F_{r-2}(x^{2s}a)} = \frac{F_{r-2}(a)}{F_{r-2}(x^{2L}a)}
\end{aligned}
\tag{5.81}
$$

and

$$
\begin{aligned}
&\frac{B(1/x^{2L-s}a)}{B(1/x^s a)} \frac{Y(x^{r(L-s)-2L-s}a)\, Y(x^{r(L-s)-2L+s}a)}{Y(1/x^{r(L-s)-s}a)\, Y(1/x^{r(L-s)+s}a)} \\
&\qquad \times \frac{S_2(1/x^{2(L-s)}a)\, G_{r-2}(1/x^{2s}a)}{S_2(1/x^{2s}a)\, G_{r-2}(1/x^{2(L-s)}a)} \\
&\qquad = \frac{G_{r-2}(1/x^{2L}a)}{G_{r-2}(1/a)}
\end{aligned}
\tag{5.82}
$$

for $|a|<1$ and $|a|>1$, respectively. Solving these recursively yields the solutions

$$
F_{r-2}(a) = \prod_{n=0}^{\infty} \tilde{F}_{r-2}(x^{2n(L-s)}a) \tag{5.83}
$$

$$
G_{r-2}(1/a) = \prod_{n=0}^{\infty} \tilde{G}_{r-2}(1/x^{2n(L-s)}a) \tag{5.84}
$$

where

$$\tilde{F}_{r-2}(a)=\frac{X(x^{r(L-s)-s}a)}{X(x^{2L-s-r(L-s)}a)}\prod_{m=0}^{\infty}\frac{A(x^{(4m+1)s}a)\,A(x^{(4m+1)s+2L}a)}{A(x^{(4m+3)s}a)\,A(x^{(4m-1)s+2L}a)}$$
$$\times\frac{R_2(x^{(4m+2)s}a)\,R_2(x^{4ms+2L}a)}{R_2(x^{(4m+4)s}a)\,R_2(x^{(4m-2)s+2L}a)} \tag{5.85}$$

and

$$\tilde{G}_{r-2}\left(\frac{1}{a}\right)=\frac{Y(1/x^{r(L-s)-s}a)}{Y(x^{r(L-s)-2L+s}/a)}\prod_{m=0}^{\infty}\frac{B(x^{(4m+1)s}/a)\,B(x^{(4m+5)s-2L}/a)}{B(x^{(4m+3)s}/a)\,B(x^{(4m+3)s-2L}/a)}$$
$$\times\frac{S_2(x^{4ms}/a)\,S_2(x^{(4m+6)s-2L}/a)}{S_2(x^{(4m+2)s}/a)\,S_2(x^{(4m+4)s-2L}/a)} \tag{5.86}$$

In the thermodynamic limit, the eigenvalue expression (2.43) for $V(z)$ is again of the form

$$V_{II}(z)=\begin{cases}V_{II}^{(l)}(z) & \text{for} \quad z>1\\ V_{II}^{(r)}(z) & \text{for} \quad z<1\end{cases} \tag{5.87}$$

only now

$$V_{II}^{(l)}(z)=-(-1)^n\,(A_{n-r}b^r)^{(L-s)/L}\,\frac{x^s z}{b}$$
$$\times\frac{A(x^{2L-s}z)\,B(1/x^{2L-s}z)}{A(x^{2s})\,B(1/x^{2s})}\,\frac{X(x^{2L+s-r(L-s)}z)}{X(x^{s+r(L-s)}z)}$$
$$\times\frac{Y(x^{r(L-s)-2L-s}/z)}{Y(1/x^{r(L-s)+s}z)}$$
$$\times\frac{R_2(x^{2L}z)\,S_2(1/x^{2L}z)\,F_{r-2}(x^{2s}z)\,G_{r-2}(1/x^{2s}z)}{R_2(x^{2s}z)\,S_2(1/x^{2s}z)\,F_{r-2}(x^{2L}z)\,G_{r-2}(1/x^{2L}z)} \tag{5.88}$$

and

$$V_{II}^{(r)}(z)=-(-1)^n\,(A_{n-r}b^r)^{(s-L)/L}\,\frac{x^s b}{z}$$
$$\times\frac{A(x^s z)\,B(1/x^s z)}{A(x^{2s})\,B(1/x^{2s})}\,\frac{X(x^{r(L-s)-s}z)}{X(x^{2L-s-r(L-s)}z)}$$
$$\times\frac{Y(1/x^{r(L-s)-s}z)}{Y(x^{r(L-s)-2L+s}/z)}$$
$$\times\frac{R_2(z)\,S_2(1/z)\,F_{r-2}(x^{2(L-s)}z)\,G_{r-2}(1/x^{2(L-s)}z)}{R_2(x^{2(L-s)}z)\,S_2(1/x^{2(L-s)}z)\,F_{r-2}(z)\,G_{r-2}(1/z)} \tag{5.89}$$

Substituting the results (5.83) and (5.84) into the expression (5.88) for $V_{II}^{(l)}(z)$, we have

$$\frac{V_{II}^{(l)}(z)}{V_0(z)} = -(A_{n-r}b^r)^{(L-s)/L}\frac{x^s z}{b}$$

$$\times\frac{X(x^{2L+s-r(L-s)}z)\;Y(x^{r(L-s)-2L-s}/z)}{X(x^{s+r(L-s)}z)\;Y(1/x^{r(L-s)+s}/z)}\frac{R_2(x^{2L}z)\;S_2(1/x^{2L}z)}{R_2(x^{2s}z)\;S_2(1/x^{2s}z)}$$

$$\times\prod_{n=0}^{\infty}\left[\frac{X(x^{s+(2n+r)(L-s)}z)\;X(x^{4L-s+(2n-r)(L-s)}z)}{X(x^{2L+s+(2n-r)(L-s)}z)\;X(x^{2L-s+(2n+r)(L-s)}z)}\right.$$

$$\times\frac{Y(1/x^{s+(2n+r)(L-s)}z)\;Y(1/x^{4L-s+(2n-r)(L-s)}z)}{Y(1/x^{2L+s+(2n-r)(L-s)}z)\;Y(1/x^{2L-s+(2n+r)(L-s)}z)}$$

$$\times\prod_{m=0}^{\infty}\frac{R_2(x^{(4m+4)s+2n(L-s)}z)\;R_2(x^{(4m+4)s+2n(L-s)+2L}z)}{R_2(x^{(4m+6)s+2n(L-s)}z)\;R_2(x^{4ms+2n(L-s)+2L}z)}$$

$$\times\frac{R_2(x^{(4m-2)s+2n(L-s)+2L}z)\;S_2(x^{(4m-2)s-2n(L-s)}/z)}{R_2(x^{4ms+2n(L-s)+4L}z)\;S_2(x^{4ms-2n(L-s)}/z)}$$

$$\times\frac{S_2(x^{(4m+4)s-2n(L-s)-2L}/z)\;S_2(x^{(4m+4)s-2n(L-s)-4L}/z)}{S_2(x^{4ms-2n(L-s)-2L}/z)\;S_2(x^{(4m+6)s-2n(L-s)-4L}/z)} \qquad (5.90)$$

This expression in turn simplifies to

$$\frac{V_{II}^{(l)}(z)}{V_0(z)} = -(A_{n-r}b^r)^{(L-s)/L}\frac{x^s z}{b}\frac{E(\tilde{a}_1/z, x^{4s})\;E(\tilde{a}_2/z, x^{4s})}{E(x^{2s}z/\tilde{a}_1, x^{4s})\;E(x^{2s}z/\tilde{a}_2, x^{4s})} \qquad (5.91)$$

The general result

$$\frac{V_{II}(w)}{V_0(w)} = \pm\frac{w}{(\tilde{a}_1\tilde{a}_2)^{1/2}}\frac{E(x^s\tilde{a}_1/w, x^{4s})\;E(x^s\tilde{a}_2/w, x^{4s})}{E(x^s w/\tilde{a}_1, x^{4s})\;E(x^s w/\tilde{a}_2, x^{4s})} \qquad (5.92)$$

follows from (5.91), the analogous result for $V_{II}^{(r)}(z)$, and the result (5.74), which again holds away from the low-temperature limit.

Returning to (5.80), we find that the equation determining the location of the excitations simplifies to

$$(A_{n-r}b^r)^{2r(L-s)/L}$$
$$=(A_{n-r}b^r)^2\frac{E(x^{r(L-s)-s}\tilde{a}_1/b,\,x^{2(L-s)})\,E(x^{r(L-s)-s}\tilde{a}_2/b,\,x^{2(L-s)})}{E(x^{r(L-s)-s}b/\tilde{a}_1,\,x^{2(L-s)})\,E(x^{r(L-s)-s}b/\tilde{a}_1,\,x^{2(L-s)})} \quad (5.93)$$

Similarly, we find that the equation for the a's is

$$a^{n-r+2}\left[\frac{E(x^s/a,\,x^{4s})}{E(x^s a,\,x^{4s})}\right]^N$$
$$+(-1)^{n-r}\,b^{2-r}(A_{n-r}b^r)^{(2s-L)/L}\frac{E(x^{r(L-s)-s}a/b,\,x^{2L-2s})}{E(x^{r(L-s)-s}b/a,\,x^{2L-2s})}$$
$$\times\prod_{m=1}^{2}\frac{E(x^{2s},\,x^{2L-2s})}{E(x^{2s}\tilde{a}_m/a,\,x^{2L-2s})}\,\mathcal{T}\left(\frac{a}{\tilde{a}_m},\,x^{2s},\,x^{2L-2s}\right)=0 \quad (5.94)$$

from which, proceeding as in Section 4, we find that the holes $\tilde{a}_1$ and $\tilde{a}_2$ satisfy

$$\left[\frac{1}{(\tilde{a}_1\tilde{a}_2)^{1/2}}\frac{E(x^s\tilde{a}_1,\,x^{4s})\,E(x^s\tilde{a}_2,\,x^{4s})}{E(x^s/\tilde{a}_1,\,x^{4s})\,E(x^s/\tilde{a}_2,\,x^{4s})}\right]^N=1 \quad (5.95)$$

indicating that the type II eigenvalues (5.92) are Nth roots of unity at $w=1$.

6. CORRELATION LENGTH

In the previous sections we have elucidated the structure of the row transfer matrix eigenvalue spectrum of the CSOS models. In general, for anisotropic interactions ($u\neq\lambda/2$ or $z\neq 1$), the eigenvalues are complex. Specifically, we have found $2(L-s)$ largest eigenvalues which are asymptotically degenerate in the thermodynamic limit. The eigenvalues corresponding to excitations from these ground states are classified into various bands of excitations. These bands contain a large number of eigenvalues and become continuous as $N\to\infty$. The largest eigenvalues are separated from the other eigenvalues by a gap which persists in the thermodynamic limit. The dominant band of excitations falls in the w band and corresponds to 1-strings. It is the existence of the gap that leads to a finite correlation length. At criticality, the eigenvalues collapse, the gap vanishes, and the correlation length diverges. In this section, we calculate the correlation length ξ by extracting the appropriate gap. This is most straightforward in the case of isotropic interactions ($u=\lambda/2$, $z=1$), since then the transfer matrix is real symmetric and the eigenvalues are real. In the anisotropic case, it is necessary to integrate[14] over the dominant band of complex eigenvalues. The results obtained by the two methods must agree, since, by general arguments,[4] the horizontal and vertical pair correlation functions and correlation length depend only on the eigenvectors of the transfer matrix. But these eigenvectors do not depend on u and so the correlation length ξ is independent of the anisotropy.

Let us consider correlations between two sites i and j in the same column on a lattice of N columns and M periodic rows. By translational invariance we can take both sites to be in column 1. Let $\boldsymbol{\varphi}_i$ be a single-site operator associated with the site at column 1 in row i with elements

$$\varphi_i(\mathbf{a}\,|\,\mathbf{b})=\varphi(a_1)\prod_{k=1}^{N}\delta(a_k,b_k) \tag{6.1}$$

The function $\varphi(a_1)$ specifies the type of correlation function; for example, to work with simple height correlations we would define this function by $\varphi(a_1)=\delta(a_1,a)$ with a some fixed height. Let V_0 be the largest eigenvalue of the transfer matrix $\mathbf{V}$ with corresponding eigenvector $|0\rangle$. The pair correlation function between two sites i and j in column 1 and separated by l rows is then given by

$$\langle\varphi_i\varphi_j\rangle=\frac{\operatorname{Tr}\,\boldsymbol{\varphi}_i\mathbf{V}^l\boldsymbol{\varphi}_j\mathbf{V}^{M-l}}{\operatorname{Tr}\,\mathbf{V}^M} \tag{6.2}$$

In the limit N, $M\to\infty$, we obtain

$$\langle\varphi_i\varphi_j\rangle=\sum_p\langle 0|\,\varphi_i\,|p\rangle\langle p|\,\varphi_j\,|0\rangle\left(\frac{V_p}{V_0}\right)^l \tag{6.3}$$

where the sum is over all eigenvalues labeled by p with the corresponding eigenvectors denoted by $|p\rangle$. For the $2(L-s)$ largest eigenvalues we have $V_p/V_0=\pm 1$, whereas for the other eigenvalues $|V_p/V_0|<1$. Hence we find

$$\langle\varphi_i\varphi_j\rangle-\lim_{i-j\to\infty}\langle\varphi_i\varphi_j\rangle={\sum_p}^{*}\langle 0|\,\varphi_i\,|p\rangle\langle p|\,\varphi_j\,|0\rangle\left(\frac{V_p}{V_0}\right)^l \tag{6.4}$$

where the starred sum excludes the $2(L-s)$ largest eigenvalues. Clearly, in the case of real eigenvalues, the decay of correlation functions for large l is determined by

$$-\xi^{-1}=\ln|V_1/V_0| \tag{6.5}$$

where V_1 is the largest eigenvalue excluding the $2(L-s)$ largest eigenvalues. This result holds provided the matrix element $\langle 0|\,\varphi_i\,|1\rangle$ does not vanish. The above arguments hold for the case of isotropic interactions ($u=\lambda/2$, $z=1$) and real eigenvalues. In the case when the transfer matrix is not real symmetric, it is necessary to integrate over the dominant band of complex eigenvalues.

6.1. Correlation Length: $2s<L$

Let us first consider the values of L and s for which $2s<L$. We discussed the nature of the elementary excitations for this case in Section 4. Specifically, we found that the leading band of eigenvalues is composed of 1-string and 2-string excitations. Our results for the corresponding eigenvalues are (4.36) and (4.73). Let us consider the isotropic case, for which all the eigenvalues are real. By solving the Bethe ansatz equations numerically and comparing the eigenvalues with the eigenspectrum obtained from direct numerical diagonalization of the transfer matrix, we find that for finite systems the leading excitation is a 1-string. In particular, the single excitation occurs at $b=-1$, that is, the excitation is located exactly at $z_n=-x^L$ for finite systems. In the thermodynamic limit, the largest of the 1-string eigenvalues has holes at $a_n=-1$ and $a_{n+1}=-1$ and is given by

$$\begin{aligned}\frac{V_1}{V_0}&=\frac{x^s}{z}\left[\frac{E(-z,x^{4s})}{E(-x^{2s}z,x^{4s})}\right]^2\\&=x^s(z^{1/2}+z^{-1/2})^2\prod_{n=1}^{\infty}\left[\frac{(1+x^{4ns}z)(1+x^{4ns}z^{-1})}{(1+x^{(4n-2)s}z)(1+x^{(4n-2)s}z^{-1})}\right]^2\end{aligned}\tag{6.6}$$

where, in the isotropic case, $z=1$. The leading 2-string excitation appears lower down in the eigenvalue spectrum. It follows that the correlation length ξ is given by

$$\begin{aligned}\xi^{-1}&=-\ln\left\{x^s\left[\frac{E(-1,x^{4s})}{E(-x^{2s},x^{4s})}\right]^2\right\}\\&=-\ln\left\{4x^s\prod_{n=1}^{\infty}\left[\frac{1+x^{4ns}}{1+x^{(4n-2)s}}\right]^4\right\}\end{aligned}\tag{6.7}$$

The above two formulas are formally identical to the corresponding results for the eight-vertex model.[4]

In the anisotropic case ($z\neq 1$) it is necessary to integrate over the band of complex 1-string eigenvalues. In this case we obtain

$$\langle \varphi_i \varphi_j \rangle - \lim_{i-j\to\infty} \langle \varphi_i \varphi_j \rangle$$

$$\sim \left(\frac{1}{2\pi i}\right)^2 \oint_{|a_n|=1} \oint_{|a_{n+1}|=1} \frac{da_n}{a_n} \frac{da_{n+1}}{a_{n+1}}$$

$$\times \rho(a_n, a_{n+1}) \left[\frac{w}{(a_n a_{n+1})^{1/2}} \frac{E(x^s a_n/w, x^{4s})\, E(x^s a_{n+1}/w, x^{4s})}{E(x^s w/a_n, x^{4s})\, E(x^s w/a_{n+1}, x^{4s})} \right]^l \tag{6.8}$$

where a_n and a_{n+1} are the locations of the two holes and the matrix elements have been absorbed into the continuous density $\rho(a_n, a_{n+1})$ of the eigenvalues. In the limit $l \to \infty$ this integral can be evaluated by steepest descents. Deforming the contours through the saddle points

$$a_n = a_{n+1} = -w/x^s \tag{6.9}$$

gives the same formula for the correlation length ξ obtained above.

To calculate the correlation length exponent, we need to revert to the original parametrization of Section 2. This yields the result

$$\xi^{-1} = -2 \ln \frac{\theta_4(0, p^{L/2s})}{\theta_4(\pi/2, p^{L/2s})} \tag{6.10}$$

where the theta function $\theta_4(u, p)$ is defined in (2.7). Near criticality we then have

$$\xi \sim \tfrac{1}{8} p^{-L/2s} \qquad \text{as} \quad p \to 0^+ \tag{6.11}$$

Hence the correlation exponent is

$$\nu = \frac{L}{2s} \tag{6.12}$$

6.2. Correlation Length: $2s > L$

In previous sections we have seen that, for $2s > L$, the excitation picture is more complicated, with the appearance of longer strings. The type I r-strings calculated in Section 5.1 are given by (5.32) with $r(L-s) < s$. The leading eigenvalue in each band of type I r-strings again occurs at $b = -1$ and is given by

$$\frac{V_I}{V_0} = x^{r(L-s)} \frac{E(-x^{s-r(L-s)}z, x^{4s})\, E(-x^{s-r(L-s)}z^{-1}, x^{4s})}{E(-x^{s+r(L-s)}z, x^{4s})\, E(-x^{s+r(L-s)}z^{-1}, x^{4s})} \tag{6.13}$$

where $z = 1$ in the isotropic case. Since the largest eigenvalue occurs for $r = 1$ in the isotropic case, this result yields the correlation length

$$\xi_I^{-1} = -\ln\left\{x^{(L-s)}\left[\frac{E(-x^{2s-L}, x^{4s})}{E(-x^L, x^{4s})}\right]^2\right\}$$
$$= -2\ln\frac{\theta_4(\pi/2 - \pi L/4s, p^{L/2s})}{\theta_4(\pi L/4s, p^{L/2s})} \tag{6.14}$$

which gives

$$\xi_I \sim \frac{1}{8\sin[\pi(L-s)/2s]}\, p^{-L/2s} \qquad \text{as} \quad p \to 0^+ \tag{6.15}$$

Hence we again find

$$\nu = \frac{L}{2s} \tag{6.16}$$

Again the correlation length ξ can be obtained by integrating over the complex band of type I 1-strings. In this case we obtain the result

$$\langle\varphi_i\varphi_j\rangle - \lim_{i-j\to\infty}\langle\varphi_i\varphi_j\rangle$$
$$\sim \frac{1}{2\pi i}\oint_{|b|=1}\frac{db}{b}\,\rho(b)\left[\frac{w}{b}\,\frac{E(x^{L-s}b/w, x^{4s})\,E(x^{L+s}w/b, x^{4s})}{E(x^{L-s}w/b, x^{4s})\,E(x^{L+s}b/w, x^{4s})}\right]^l \tag{6.17}$$

The saddle point now occurs at

$$b = -\frac{w}{x^s} \tag{6.18}$$

so again the method of steepest descents gives the same formula for the correlation length ξ as given above.

7. CONCLUSION

In this paper we have calculated the free energy and the largest bands of eigenvalues of the row transfer matrix for the cyclic solid-on-solid models with L heights and crossing parameter $\lambda = \pi s/L$. The Wiener–Hopf perturbation methods applied can be straightforwardly extended to obtain bands of eigenvalues further down in the spectrum. We have found two different classifications of the string excitations applying to the cases $2s < L$ and $2s > L$. These agree with the classification of excitations for the eight-vertex model proposed by Klümper and Zittartz.[15,16] From the dominant band of excitations we have obtained formulas for the correlation length. The associated critical exponent is

$$\nu = L/2s \tag{7.1}$$

in agreement with the scaling relations. This is the central result of this paper. In particular, for the three-coloring problem,[5] we notice that the correlation length ξ is given by (6.14) with $L=3$ and $s=2$, so that in this case $\nu = 3/4$.

There is further work to be done. It remains to obtain interfacial tensions by calculating the asymptotic degeneracy of the largest eigenvalues. It would also be of interest to solve the inversion identity (2.33) directly by the methods of this paper. In the thermodynamic limit this equation simplifies considerably, since the second term on the right side is exponentially small in the strip $-\lambda/2 < u < \lambda/2$. It therefore follows that, in the thermodynamic limit, all eigenvalues must satisfy the functional relation or inversion relation

$$V(u)\, V(u+\lambda) = \phi(\lambda - u)\, \phi(\lambda + u) \tag{7.2}$$

in this strip. Following Klümper and Zittartz, let us define excitation functions $l(u)$ by

$$l(u) = \lim_{N \to \infty} \frac{V(u)}{V_0(u)} \tag{7.3}$$

Then the excitation functions satisfy the functional relation

$$l(u)\, l(u+\lambda) = 1 \tag{7.4}$$

or, in terms of z,

$$l(z)\, l(x^{2s} z) = 1 \tag{7.5}$$

Finally, it is indeed verified directly that all of the expressions obtained for the various bands of eigenvalues in this paper satisfy this functional relation.

APPENDIX A. INVERSION IDENTITY DERIVATION

In this Appendix we derive[26] the inversion identity (2.33).

In writing down the vertex weights (2.1)–(2.4), we have suppressed an explicit dependence on gauge factors,[2] which in any case cancel out of the transfer matrix for periodic boundary conditions. However, in deriving the inversion identity, it is convenient to make a specific choice of gauge. In particular, we make a choice that destroys the diagonal reflection symmetry in the weights β_a, as given in (2.2), which is now replaced by

$$W\begin{pmatrix} a & a-1 \\ a+1 & a \end{pmatrix} = -\frac{\theta_1(u)\,\theta_4(w_{a-1})}{\theta_1(\lambda)\,\theta_4(w_a)} \tag{A.1}$$

$$W\begin{pmatrix} a & a+1 \\ a-1 & a \end{pmatrix} = -\frac{\theta_1(u)\,\theta_4(w_{a+1})}{\theta_1(\lambda)\,\theta_4(w_a)} \tag{A.2}$$

The remaining weights are unchanged.

The elements of the commuting family of transfer matrices $\mathbf{V}(u)$ are gauge invariant and given in terms of the face weights by

$$V(u)_{aa'} = \prod_{j=1}^{N} W\left(\begin{matrix} a'_j & a'_{j+1} \\ a_j & a_{j+1} \end{matrix}\,\middle|\, u\right) \tag{A.3}$$

We define a matrix $\mathbf{R}$ by

$$R\begin{pmatrix} d & c \\ a & b \end{pmatrix}_{fg} = W\left(\begin{matrix} f & g \\ a & b \end{matrix}\,\middle|\, u\right) W\left(\begin{matrix} d & c \\ f & g \end{matrix}\,\middle|\, u+\lambda\right) \tag{A.4}$$

In the symmetric gauge the usual local inversion relation takes the form

$$\sum_g R\begin{pmatrix} d & c \\ b & c \end{pmatrix}_{ag} = \frac{\theta_1(\lambda-u)\,\theta_1(\lambda+u)}{\theta_1(\lambda)^2}\,\delta(b,d) \tag{A.5}$$

where $\delta(b, d)$ is a Kronecker delta function. With periodic boundary conditions it follows that

$$[V(u)\,V(u+\lambda)]_{aa'} = \operatorname{Tr} \prod_{j=1}^{N} R\begin{pmatrix} a'_j & a'_{j+1} \\ a_j & a_{j+1} \end{pmatrix} \tag{A.6}$$

where the product is an ordered product of matrices and N must be even. The various types of $\mathbf{R}$ matrices are given explicitly as follows:

Diagonal (2×2 matrices)

$$R\begin{pmatrix} a-1 & a \\ a-1 & a \end{pmatrix} = \frac{\theta_4(w_a)}{\theta_4(w_{a-1})}\begin{pmatrix} -\dfrac{\theta_1(\lambda-u)\,\theta_1(\lambda+u)}{\theta_1(\lambda)^2} & 0 \\ \dfrac{\theta_4(w_a-u)\,\theta_4(w_a+u)}{\theta_4(w_a)^2} & \dfrac{\theta_1(u)^2\,\theta_4(w_{a-1})\,\theta_4(w_{a+1})}{\theta_1(\lambda)^2\,\theta_4(w_a)^2} \end{pmatrix} \tag{A.7}$$

$$R\begin{pmatrix} a+1 & a \\ a+1 & a \end{pmatrix}$$

$$= \frac{\theta_4(w_a)}{\theta_4(w_{a+1})} \begin{pmatrix} \dfrac{\theta_1(u)^2\,\theta_4(w_{a-1})\,\theta_4(w_{a+1})}{\theta_1(\lambda)^2\,\theta_4(w_a)^2} & \dfrac{\theta_4(w_a-u)\,\theta_4(w_a+u)}{\theta_4(w_a)^2} \\ 0 & -\dfrac{\theta_1(\lambda-u)\,\theta_1(\lambda+u)}{\theta_1(\lambda)^2} \end{pmatrix} \tag{A.8}$$

Mixed (2×1 and 1×2 vectors)

$$R\begin{pmatrix} a & a-1 \\ a & a+1 \end{pmatrix} = \frac{\theta_4(w_{a-1})}{\theta_4(w_a)} \begin{pmatrix} \dfrac{\theta_1(\lambda-u)}{\theta_1(\lambda)}\dfrac{\theta_4(w_{a-1}-u)}{\theta_4(w_{a-1})} \\ -\dfrac{\theta_1(\lambda+u)}{\theta_1(\lambda)}\dfrac{\theta_4(w_{a+1}-u)}{\theta_4(w_{a+1})} \end{pmatrix} \tag{A.9}$$

$$R\begin{pmatrix} a & a+1 \\ a & a-1 \end{pmatrix} = \frac{\theta_4(w_{a+1})}{\theta_4(w_a)} \begin{pmatrix} \dfrac{\theta_1(\lambda+u)}{\theta_1(\lambda)}\dfrac{\theta_4(w_{a-1}+u)}{\theta_4(w_{a-1})} \\ -\dfrac{\theta_1(\lambda-u)}{\theta_1(\lambda)}\dfrac{\theta_4(w_{a+1}+u)}{\theta_4(w_{a+1})} \end{pmatrix} \tag{A.10}$$

$$R\begin{pmatrix} a-1 & a \\ a+1 & a \end{pmatrix} = -\frac{\theta_1(u)\,\theta_4(w_a+u)}{\theta_1(\lambda)\,\theta_4(w_a)}\begin{pmatrix} 1 & 1 \end{pmatrix} \tag{A.11}$$

$$R\begin{pmatrix} a+1 & a \\ a-1 & a \end{pmatrix} = -\frac{\theta_1(u)\,\theta_4(w_a-u)}{\theta_1(\lambda)\,\theta_4(w_a)}\begin{pmatrix} 1 & 1 \end{pmatrix} \tag{A.12}$$

Nondiagonal (1×1 scalars)

$$R\begin{pmatrix} a-1 & a \\ a+1 & a+2 \end{pmatrix} = R\begin{pmatrix} a+1 & a \\ a-1 & a-2 \end{pmatrix} = -\frac{\theta_1(u)\,\theta_1(\lambda-u)}{\theta_1(\lambda)^2} \tag{A.13}$$

$$R\begin{pmatrix} a & a-1 \\ a+2 & a+1 \end{pmatrix} = \frac{\theta_1(u)\,\theta_1(\lambda+u)}{\theta_1(\lambda)^2}\frac{\theta_4(w_{a-1})}{\theta_4(w_{a+1})} \tag{A.14}$$

$$R\begin{pmatrix} a & a+1 \\ a-2 & a-1 \end{pmatrix} = \frac{\theta_1(u)\,\theta_1(\lambda+u)}{\theta_1(\lambda)^2}\frac{\theta_4(w_{a+1})}{\theta_4(w_{a-1})} \tag{A.15}$$

The eigenvalues of the 2×2 matrices are determined by some simple left and right eigenvectors according to the equations

$$\begin{pmatrix} 1 & 1 \end{pmatrix} R\begin{pmatrix} a-1 & a \\ a-1 & a \end{pmatrix} = \frac{\theta_1(u)^2\,\theta_4(w_{a+1})}{\theta_1(\lambda)^2\,\theta_4(w_a)}\begin{pmatrix} 1 & 1 \end{pmatrix} \tag{A.16}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} R\begin{pmatrix} a+1 & a \\ a+1 & a \end{pmatrix} = \frac{\theta_1(u)^2\,\theta_4(w_{a-1})}{\theta_1(\lambda)^2\,\theta_4(w_a)}\begin{pmatrix} 1 & 1 \end{pmatrix} \tag{A.17}$$

$$R\begin{pmatrix} a-1 & a \\ a-1 & a \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\theta_1(\lambda-u)\,\theta_1(\lambda+u)\,\theta_4(w_a)}{\theta_1(\lambda)^2\,\theta_4(w_{a-1})}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (A.18)$$

$$R\begin{pmatrix} a+1 & a \\ a+1 & a \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\theta_1(\lambda-u)\,\theta_1(\lambda+u)\,\theta_4(w_a)}{\theta_1(\lambda)^2\,\theta_4(w_{a+1})}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (A.19)$$

which follow readily from the elliptic function identity

$$\begin{aligned} &\theta_4(w_a+u)\,\theta_4(w_a-u) \\ &\quad = \theta_1(u)^2\,\theta_4(w_{a-1})\,\theta_4(w_{a+1}) + \theta_1(\lambda-u)\,\theta_1(\lambda+u)\,\theta_4(w_a)^2 \end{aligned} \quad (A.20)$$

Other properties of the **R** matrices needed are the inner products

$$\begin{aligned} &(1\quad 1)\begin{pmatrix} a & a-1 \\ a & a+1 \end{pmatrix} \\ &\quad = \frac{\theta_4(w_{a-1})}{\theta_4(w_a)}\left[\frac{\theta_1(\lambda-u)\,\theta_4(w_{a-1}-u)}{\theta_1(\lambda)\,\theta_4(w_{a-1})} - \frac{\theta_1(\lambda+u)\,\theta_4(w_{a+1}-u)}{\theta_1(\lambda)\,\theta_4(w_{a+1})}\right] \\ &\quad = -\frac{\theta_1(u)\,\theta_1(2\lambda)\,\theta_4(w_a-u)}{\theta_1(\lambda)^2\,\theta_4(w_{a+1})} \end{aligned} \quad (A.21)$$

and

$$\begin{aligned} &(1\quad 1)\,R\begin{pmatrix} a & a+1 \\ a & a-1 \end{pmatrix} \\ &\quad = -\frac{\theta_4(w_{a+1})}{\theta_4(w_a)}\left[\frac{\theta_1(\lambda+u)\,\theta_4(w_{a-1}+u)}{\theta_1(\lambda)\,\theta_4(w_{a-1})} - \frac{\theta_1(\lambda-u)\,\theta_4(w_{a+1}+u)}{\theta_1(\lambda)\,\theta_4(w_{a+1})}\right] \\ &\quad = -\frac{\theta_1(u)\,\theta_1(2\lambda)\,\theta_4(w_a+u)}{\theta_1(\lambda)^2\,\theta_4(w_{a-1})} \end{aligned} \quad (A.22)$$

The elements of the matrix $\mathbf{V}(u)\,\mathbf{V}(u+\lambda)$ fall into three categories. They are completely diagonal, completely nondiagonal, or mixed. In all cases we find that the matrix elements satisfy the inversion identity

$$\mathbf{V}(u)\,\mathbf{V}(u+\lambda) = \left[\frac{\theta_1(\lambda-u)\,\theta_1(\lambda+u)}{\theta_1(\lambda)^2}\right]^N \mathbf{I} + \left[\frac{\theta_1(u)}{\theta_1(\lambda)}\right]^N \mathbf{P}(u) \quad (A.23)$$

where **I** is the identity matrix and $\mathbf{P}(u)$ is an auxiliary matrix that commutes with $\mathbf{V}(u)$ and whose elements are entire functions of u. More specifically, if an element of $\mathbf{V}(u)\,\mathbf{V}(u+\lambda)$ is completely nondiagonal, then each **R** matrix is a scalar and each contributes a factor $\theta_1(u)/\theta_1(\lambda)$, so the element is of the form of the second term in the inversion identity. If an element falls in the mixed category, then it breaks up into a number of scalar segments, starting and ending at nondiagonal points. But the 1×2 mixed **R** matrices are left eigenvectors of the 2×2 diagonal **R** matrices. Hence they can be propagated to the right. Each diagonal 2×2 **R** matrix thus contributes a factor $\theta_1(u)/\theta_1(\lambda)$, which comes from the eigenvalue of its left eigenvector. The remaining inner product of **R** matrices contributes two factors of $\theta_1(u)/\theta_1(\lambda)$, one from the 1×2 mixed **R** matrix itself and the other from the inner product as discussed above. So again such elements must be of the form of the second term in the inversion identity. Finally, if an element is completely diagonal, then the prefactors of the 2×2 diagonal **R** matrices, which are just gauge factors, cancel out and the trace is given by the sum of the Nth powers of the common eigenvalues. Clearly this is of the required form, where now both terms on the right-side of the inversion identity are needed.

The inversion identity is in fact just the first functional equation in a fusion hierarchy[28,29] relating solvable models constructed by fusing together[30–34] $p \times q$ blocks of elementary faces. This construction gives rise[29] to commuting transfer matrices $\mathbf{V}^{pq}(u)$ satisfying

$$V^{pq}(u)\, V^{pr}(v) = V^{pr}(v)\, V^{pq}(u) \tag{A.24}$$

In this way the auxiliary matrix $\mathbf{P}(u)$ in the CSOS inversion identity is identified as the row transfer matrix of the solvable model resulting from 1×2 fusion.

APPENDIX B. RESHETIKHIN'S ANALYTIC ANSATZ

In this Appendix we derive the Bethe ansatz equations from the inversion identity (2.33) and the properties (2.36)–(2.38), again with $N = 0$ (mod L).

Operating with either side of the inversion identity on a fixed common eigenvector of $\mathbf{V}(u)$ and $\mathbf{P}(u)$ gives a set of functional equations for the eigenvalues of the transfer matrix of the same form, namely

$$V(u)\, V(u+\lambda) = \phi(\lambda - u)\, \phi(\lambda + u) + \phi(u)\, P(u) \tag{B.1}$$

$$V^*(u) = V(\lambda - u) \tag{B.2}$$

$$V(u+\pi) = (-1)^N\, V(u) \tag{B.3}$$

$$V(u+\pi\tau) = (-p^{-1} e^{-2iu})^N\, e^{2ni\lambda} V(u) \tag{B.4}$$

We begin in the $n=0$ sector, where all the arrows are up and $\mathbf{V}(u)$ is the simple cyclic $L \times L$ matrix

$$\mathbf{V}(u)=\begin{pmatrix} 0 & a & 0 & \cdots & 0 & b \\ b & 0 & a & \cdots & 0 & 0 \\ 0 & b & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a \\ a & 0 & 0 & \cdots & b & 0 \end{pmatrix} \tag{B.5}$$

in which $a=\phi(\lambda-u)$ and $b=\phi(u)$. The eigenvalues are given by

$$V(u)=\omega\phi(\lambda-u)+\omega^{-1}\phi(u) \tag{B.6}$$

where $\omega^L=1$. This $V(u)$ is seen to satisfy (B.1)–(B.4).

Now, following Reshetikhin,[27] we use the ansatz

$$V(u)=\phi(\lambda-u)\,A(u)+\phi(u)\,B(u) \tag{B.7}$$

for the $n \geqslant 1$ cases. The functions $A(u)$ and $B(u)$ are to be quasiperiodic, meromorphic, and each have the same number of zeros and poles. Since $V(u)$ has no poles, the poles of $A(u)$ must be the same as the poles of $B(u)$ and their residues must cancel, that is,

$$A(u)=C\prod_{j=1}^{n}\frac{\theta_1(u-a_j)}{\theta_1(u-s_j)}, \qquad B(u)=D\prod_{j=1}^{n}\frac{\theta_1(u-b_j)}{\theta_1(u-s_j)} \tag{B.8}$$

with

$$C\phi(\lambda-s_k)\prod_{j=1}^{n}\theta_1(s_k-a_j)+D\phi(s_k)\prod_{j=1}^{n}\theta_1(s_k-b_j)=0 \tag{B.9}$$

In order to satisfy the inversion identity, we see that substitution of the ansatz (B.7) into (B.1) requires that

$$A(u)\,A^*(-u)=1 \tag{B.10}$$

$$B(u)=A^*(\lambda-u) \tag{B.11}$$

Now these last two equations are to be solved for a_j, b_j, C and D in terms of s_j. The result (B.9) then becomes the Bethe ansatz equation determining s_k. Equating zeros and poles in (B.11) yields

$$b_j=\lambda-a_j^*, \qquad s_j=\lambda-s_j^* \tag{B.12}$$

Similarly, from (B.10) we have

$$s_j = -a_j^*, \qquad a_j = -s_j^* \tag{B.13}$$

Substitution of these results into (B.9) then yields

$$\frac{\phi(\lambda - s_k)}{\phi(s_k)} = -\prod_{j=1}^{n} \frac{\theta_1(s_k - s_j - \lambda)}{\theta_1(s_k - s_j + \lambda)} \tag{B.14}$$

for $k = 1, \ldots, n$.

The eigenvalues are given by

$$V(u) = \phi(\lambda - u) \prod_{j=1}^{n} \frac{\theta_1(u - s_j + \lambda)}{\theta_1(u - s_j)} + \phi(u) \prod_{j=1}^{n} \frac{\theta_1(u - s_j - \lambda)}{\theta_1(u - s_j)} \tag{B.15}$$

These equations can in turn be written in the form

$$V(u)\, Q(u) = \phi(\lambda - u)\, Q(u + \lambda) + \phi(u)\, Q(u - \lambda) \tag{B.16}$$

where

$$Q(u) = \prod_{j=1}^{n} \theta_1(u - s_j) \tag{B.17}$$

The functions $Q(u)$ and $\phi(u)$ obey the quasiperiodicity conditions

$$Q(u + \pi) = (-1)^n\, Q(u) \tag{B.18}$$

$$Q(u + \pi\tau) = (-p^{-1} e^{-2iu})^n\, Q(u) \prod_{j=1}^{n} e^{2is_j} \tag{B.19}$$

$$\phi(u + \pi) = (-1)^N\, \phi(u) \tag{B.20}$$

$$\phi(u + \pi\tau) = (-p^{-1} e^{-2iu})^N\, \phi(u) \tag{B.21}$$

which are consistent with the conditions (B.3) and (B.4).

APPENDIX C. PROOF OF SOME TYPICAL IDENTITIES

In this Appendix we give some details of the calculation of the simplified forms for the expressions $V_0^{(l)}(z)$ and $V_0^{(r)}(z)$ in (3.20) and (3.21).

For the functions defined in (3.8), we begin by noting that

$$N^{-1} \log A(z) = -\sum_{k=1}^{\infty} \frac{z^k}{k(1 - x^{2Lk})} \tag{C.1}$$

$$N^{-1} \log B\left(\frac{1}{z}\right) = -\sum_{k=1}^{\infty} \frac{x^{2Lk}}{kz^k(1 - x^{2Lk})} \tag{C.2}$$

Consider first the expression for $V_0^{(l)}(z)$. Using the above results, we expand $-N^{-1}\log V_0^{(l)}(z)$ and collect the terms which go as z^k:

$$\sum_{k=1}^{\infty}\frac{z^k}{k(1-x^{2Lk})}\left\{x^{(2L-s)k}+\sum_{m=0}^{\infty}x^{4msk}[x^{3sk}-x^{5sk}+x^{(2L+3s)k}-x^{(2L+s)k}]\right\}$$

$$=\sum_{k=1}^{\infty}\frac{z^k}{k(1-x^{2Lk})(1-x^{4sk})}[x^{3sk}-x^{5sk}+x^{(2L-s)k}-x^{(2L+s)k}] \quad \text{(C.3)}$$

These in turn can be written as

$$\sum_{k=1}^{\infty}\frac{w^k(x^{2sk}+x^{2(L-s)k})}{k(1-x^{2Lk})(1+x^{2sk})} \quad \text{(C.4)}$$

where we have also made the substitution $w=x^s z$. In a similar manner the z^{-k} terms are given by

$$\sum_{k=1}^{\infty}\frac{x^{2Lk}}{kz^k(1-x^{2Lk})}\left\{x^{(s-2L)k}+\sum_{m=0}^{\infty}x^{4msk}[x^{-sk}-x^{sk}+x^{(3s-2L)k}-x^{(s-2L)k}]\right\} \quad \text{(C.5)}$$

$$=\sum_{k=1}^{\infty}\frac{x^{2Lk}}{kz^k(1-x^{2Lk})(1-x^{4sk})}[x^{-sk}-x^{sk}+x^{(3s-2L)k}-x^{(5s-2L)k}] \quad \text{(C.6)}$$

$$=\sum_{k=1}^{\infty}\frac{x^{2sk}(x^{2sk}+x^{2(L-s)k})}{kw^k(1-x^{2Lk})(1+x^{2sk})} \quad \text{(C.7)}$$

Finally we need to consider the constant term [i.e., the z^0 terms in (3.20)], where the expansion gives

$$-\sum_{k=1}^{\infty}\frac{x^{2sk}+x^{2(L-s)k}}{k(1-x^{2Lk})} \quad \text{(C.8)}$$

Adding the three contributions (C.4), (C.7), and (C.8) gives the result (3.22). The same result is obtained from $V_0^{(r)}(z)$ in (3.21) by similar arguments.

In generalizing these arguments to more complicated identities, we also make use of the Taylor expansion of the elliptic function (2.10):

$$\log E(z,x)=-\sum_{k=1}^{\infty}\frac{z^k+x^k(1+z^{-k})}{k(1-x^k)} \quad \text{(C.9)}$$

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